Lecture 38 Quadrature domains (Chap 22, 23)

Ave property $\Rightarrow$ discs are "quadrature domains"

$$h(a) = \frac{1}{\text{Area}(D_r(a))} \int_{D_r(a)} h(z)\, dA_z$$

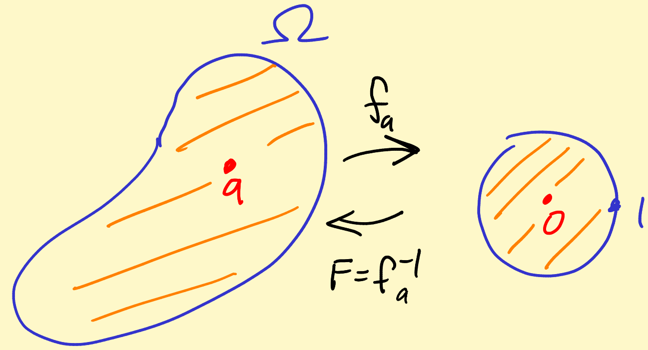
One point quad. identity of order 0.

$h \in H^2(D_r(a))$

So $h(a) = \langle h, K_a \rangle$ where $K_a(z) \equiv \dfrac{1}{\text{Area}(D_r(a))}$

Fact (Epstein) Discs are only domains with one point q.i.'s of order 0.

Pf in simply connected case



Suppose $\iint_\Omega h\, dA = \langle h, 1 \rangle = c\, h(a)$ ← q.i.

for all $h \in H^2(\Omega)$.

[a is a fixed pt and c does not depend on h.]

Let $h \equiv 1$ to see $c = A(\Omega)$ ← area of Ω

Claim $K_a \equiv \dfrac{1}{A(\Omega)}$ ← analytic, represents fcn$^\ell$ $h \mapsto h(a)$ ✓

$$K_{D_1(0)}(z,w) = \frac{1}{\pi(1-z\bar{w})^2} \qquad K = \text{kernel for } \Omega.$$

Transformation formula for Bergman kernels:

$$\underbrace{f_a'(z)\, K_{D_1(0)}(f_a(z), 0)}_{\frac{1}{\pi}} = \underbrace{K(z, \underbrace{F(0)}_{a})}_{K_a(z) = \frac{1}{A(\Omega)}} \underbrace{\overline{F'(0)}}_{\text{constant}}$$

Aha! $f_a'(z) \equiv \text{const} = A$

So $f_a(z) = Az + B$

So F is linear too, and $\Omega = \text{disc}$.

[Pf that Ω cannot have holes not hard.]

Fun thing: Look for Ω that are two-point q.d. of order 0. S.C. case: Get Riemann map mapping one pt to $0 \in D_1(0)$. Play around with transf formula. Get Ω = "Neumann ovals"

Def^n Suppose Ω is a domain of bounded area such that

$$\boxed{\iint_\Omega h\, dA = \sum_{n=1}^{N} \sum_{k=0}^{M_n} c_{nk}\, h^{k}(a_n)} \qquad \text{Q.I.} \ (*)$$

for all $h \in H^2(\Omega)$, where $\{a_n\}_{n=1}^N$ are fixed pts in Ω and $c_{nk} \in \mathbb{C}$ do not depend on h.

Ω is a "quadrature domain" (q.d.)

[N-point q.i. of order $M = \underset{n=1,\dots,N}{\mathrm{Max}}(M_n)$]

Simply connected case.

$$K(z,w) = f'(z)\, K_{D_1(0)}\big(f(z), f(w)\big)\, \overline{f'(w)}$$

$$= \frac{f'(z)\,\overline{f'(w)}}{\pi\,\big(1 - f(z)\overline{f(w)}\big)^2}$$

see $K(z,w)$ analytic in z, anti-holo in w.

Notation $\quad K_a^m(z) = \left.\dfrac{\partial^m}{\partial \bar{w}^m} K(z,w)\right|_{w=a}$

convention: $K_a^0(z) = K_a(z) = K(z,a)$.

$$h(a) = \langle h, K_a \rangle = \iint_\Omega h(z)\, \overline{K(z,a)}\, dA$$

$h^{(m)}(a) = \langle h, K_a^{(m)} \rangle \leftarrow$ differentiate in a under $\iint$ and use formula for $K(z,w)$.

Better way (see Chap 22). θ_a radially symmetric real valued C_0^∞ bump fcn supported on small disc about a.

Know $\quad h(a) = \langle h, \theta_a \rangle$ $\quad \left[K_a = B\theta_a \right]$

diff this wrt a

$$h^{(m)}(a) = \iint h \underbrace{\left[\frac{\partial^m}{\partial a^m} \theta_a \right]}_{\theta_a^{(m)} \in C_0^\infty(\Omega)} dA$$

m-th deriv fcn$^\ell$ at a : $h \longmapsto h^{(m)}(a)$

is a cont. lin. fcn$^\ell$ $\quad h^{(m)}(a) = \langle h, \theta_a^{(m)} \rangle$

represented by $\quad K_a^{(m)}(z) = \left(B\,\theta_a^{(m)} \right)(z)$

Trick $\quad K_a^{(m)}(z) = \langle K_a^{(m)}, K_z \rangle$

$$= \overline{\langle K_z, K_a^{(m)} \rangle}$$

$$= \overline{\left. \frac{\partial^m}{\partial w^m} K_z(w) \right|_{w=a}}$$

$$= \left. \frac{\partial^m}{\partial \bar{w}^m} K(w,z) \right|_{w=a}$$

$$= \frac{\partial^m}{\partial \bar{w}^m} K(z,w)\Big|_{w=a} = K_a^{(m)}(z)$$

<u>Easy fact</u> Ω satisfies a Q.D. $(*)$ $\iff$

$$\underbrace{1 \equiv \sum_{n=1}^{N} \sum_{m=0}^{M_n} \overline{c_{nm}}\, K_{a_n}^{(m)}}_{\text{element of "Bergman span"}}$$

<u>Pf</u> $\iint_\Omega h \, dA = \langle h, 1 \rangle$

Claim $= \langle h, \kappa \rangle$

$$= \sum_{n=1}^{N} \sum_{m=0}^{M_n} \langle h, \overline{c_{nm}}\, K_{a_n}^{(m)} \rangle$$

$$= \sum_{n=1}^{N} \sum_{m=0}^{M_n} c_{nm}\, h^{(m)}(a_n) \leftarrow \text{q.i.}$$

So $\langle h, \underbrace{1-\kappa}_{} \rangle = 0$ for <u>all</u> $h \in H^2(\Omega)$.

$\in H^2(\Omega) \perp$ to $H^2(\Omega) \Rightarrow 1 - \kappa \equiv 0$.

<u>Next time</u> A simply connected domain $\Omega \neq \mathbb{C}$ is a q.d. $\iff$ inverse of Riemann map is a <u>rational fcn</u>!