Lecture 39 Quadrature domains & the Bergman kernel

$$\iint_{\Omega} h \, dA = \sum_{n=1}^{N} \sum_{m=0}^{M_n} c_{nm} h^{(m)}(a_n) \leftarrow \text{q.i.}$$

<u>Thm</u>: (Avci) Suppose $f : \Omega_1 \to \Omega_2$ is a biholomorphic map between domains.
Ω_2 is a quadrature domain (q.d.) $\Longleftrightarrow$
f' belongs to the "Bergman span" of Ω_1
(= complex linear span $\mathcal{K}_1$ of

$$\{ K_1^{(m)}(\cdot, a) \; ; \; a \in \Omega, \; m = 0, 1, 2, \ldots \}$$

where $K_1^{(m)}(z,a) = K_{1_a}^{(m)}(z) = \frac{\partial^m}{\partial \bar{w}^m} K(z,w) \Big|_{w=a}$)

<u>Pf</u>:

$$\iint_{\Omega_2} h \, dA = \iint_{\Omega_1} \underbrace{|f'|^2}_{=|\text{Real Jacobian}|} (h \circ f) \, dA$$

$$= \iint_{\Omega_1} f'(h \circ f) \underbrace{\overline{f'}}_{f' = \sum c_{mn} K_{1_{a_n}}^{(m)}} \, dA$$

$$= \sum \overline{c_{nm}} \langle f'(h \circ f), K_{1_{a_n}}^{(m)} \rangle$$

$$= \sum \overline{c_{nm}} \frac{\partial^m}{\partial z^m}\left[f'(h \circ f) \right] \Big|_{z=a_n}$$

$= q.i. =$ finite linear combo of values and derivatives of h at points in $\{f(a_n) : n=1,\dots,N\}$

Proof of $(\Leftarrow)$ is done. Easy.

Proof of $(\Rightarrow)$: Hmmm. $\langle h, f' \rangle_{\Omega_1} =$

$$\langle h, \underbrace{f'(1 \circ f)}_{\Lambda_1(1)} \rangle_{\Omega_1} = \langle \Lambda_2 h, 1 \rangle_{\Omega_2}$$

where $\Lambda_2 h = F'(h \circ F)$ where $F = f^{-1}$.

Aha! $\cdots = \iint_{\Omega_2} F'(h \circ F)\, dA$.

If Ω_2 is a q.d. $\cdots =$ lin combo of values and derivatives at $\{a_n\}_{n=1}^N$ in Ω_2.

$=$ lin combo of values and deriv of h at $\{F(a_n)\}_{n=1}^N$ in Ω.

$= \langle h, \kappa \rangle$ for the appropriately written $\kappa \in \mathcal{K}_1$.

[see pf that q.i. equiv to $1 \in \mathcal{K}$]

Aha! $\langle h, f' \rangle = \langle h, \kappa \rangle$ for all $h \in H^2(\Omega_1)$.

$\Rightarrow f' = \kappa$. So $f' \in \mathcal{K}_1$. ✓

Important application If $\Omega \neq \mathbb{C}$ and simply connected, Riemann map $f: \Omega \to D_1(0)$, and Thm $\Rightarrow$ $\left[\Omega \text{ is a q.d.} \iff F' \in \mathcal{K}_{D_1(0)} \right]$

where $F = f^{-1}$.

Bergman span of unit disc $K(z,w) = K_{D_1(0)}(z,w)$.

$$K(z,w) = \frac{1}{\pi(1-z\bar{w})^2} \qquad K_0^{(0)}(z) = \frac{1}{\pi}$$

$$K^{(1)}(z,w) = \frac{(-2)(-z)}{\pi(1-z\bar{w})^3} = \frac{2z}{\pi(1-z\bar{w})^3} \qquad K_0^{(1)}(z) = \frac{2}{\pi} z$$

$$K^{(2)}(z,w) = \frac{3\cdot 2\, z^2}{\pi(1-z\bar{w})^4} \qquad K_0^{(2)} = \frac{3!}{\pi} z^2$$

⋮

$$K^{(m)}(z,w) = \frac{(m+1)!\, z^m}{\pi(1-z\bar{w})^{m+2}}$$

⋮

$$K_0^{(m)}(z) = \frac{(m+1)!}{\pi} z^{m+2}$$

Hmmm. Rational fcns! Poles at ?

$$(1-z\bar{w}) = -\bar{w}\left(z - \frac{1}{\bar{w}}\right)$$

pole at $\frac{1}{\bar{w}}$ which is outside $C_1(0)$ when $w \in D_1(0)$.

And powers 2 or more $\Rightarrow$ all residues are zero.

Bergman span of $D_1(0)$ = All rational fcns (including polynomials) with residue free poles outside $\overline{D_1(0)}$.

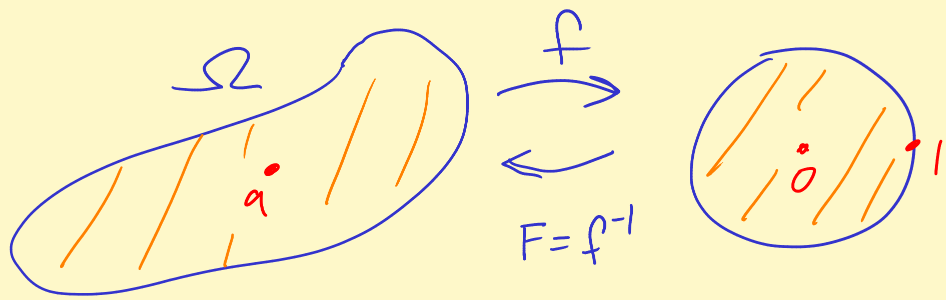
If f' is in this space, easy to write down a rational antiderivative to see

Thm: A simply connected domain $\Omega \neq \mathbb{C}$ is a q.d. $\Longleftrightarrow$ the inverse of a Riemann map is rational.

Cor: Non-trivial q.d. exist!

In fact, q.d. are <u>dense</u> in a strong sense.

Play around



Suppose Ω has smooth bndry. Then $F \in C^\infty(\overline{D_1(0)})$

Aha! Get $F(rz)$ <u>close</u> to F in strong sense of $C^\infty(\overline{D_1(0)})$ when $r \nearrow 1$. $F(rz)$ analytic on an open set containing $\overline{D_1(0)}$. Get Taylor poly $P(z)$ close to $F(rz)$ in C^∞ sense.

P is in $\mathcal{K}_{D_1(0)}$! (Close enough so P is biholo.)

Exciting consequence: Found a q.d. $\tilde{\Omega} = P(D_1(0))$ that is C^∞ close to Ω.