Lecture 40 Quadrature domains & the Schwarz fcn

Bergman span of unit disc

$K(z,w) = K_{D_1(0)}(z,w)$.

$$K(z,w) = \frac{1}{\pi(1-z\bar{w})^2} = \frac{1}{\pi \bar{w}^2 (z - 1/\bar{w})^2} \qquad K_0^{(0)}(z) = \frac{1}{\pi}$$

$$K^{(1)}(z,w) = \frac{2\overset{(z-1/\bar{w})+1/\bar{w}}{z}}{\pi(1-z\bar{w})^3} = \frac{2}{-\pi\bar{w}^3(z-1/\bar{w})^2} - \frac{2}{\pi\bar{w}^4(z-1/\bar{w})^3} \qquad K_0^{(1)}(z) = \frac{2}{\pi} z$$

$$K^{(2)}(z,w) = \frac{3\cdot 2\, z^2}{\pi(1-z\bar{w})^4} = \text{etc.} \qquad K_0^{(2)} = \frac{3!}{\pi} z^2$$

$$\vdots \qquad\qquad \vdots$$

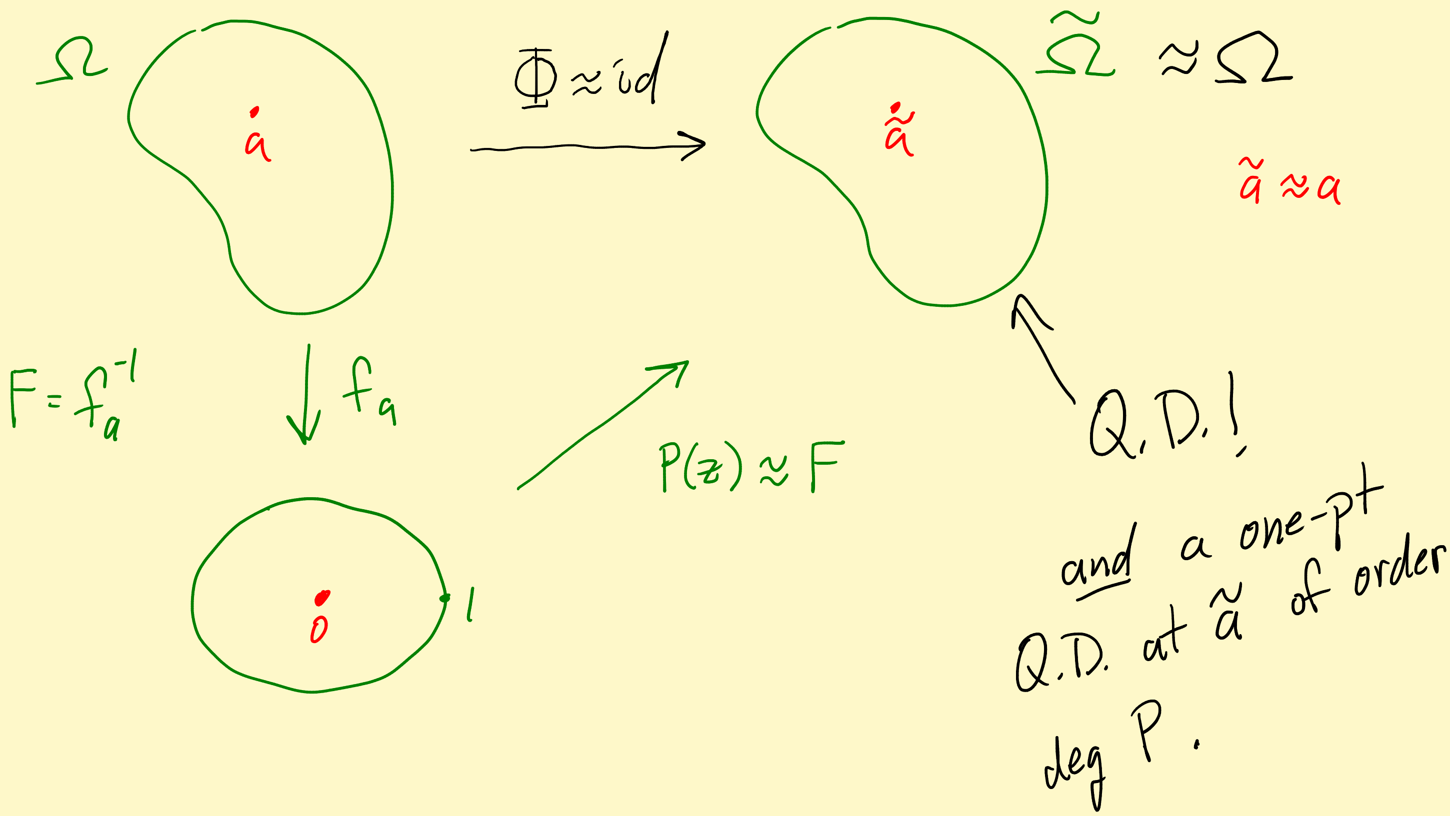
$$K^{(m)}(z,w) = \frac{(m+1)!\, z^m}{\pi(1-z\bar{w})^{m+2}} \qquad K_0^{(m)}(z) = \frac{(m+1)!}{\pi} z^{m+2}$$

Fact: Bergman span of unit disc is $\mathcal{K} =$ rational fcns with residue free poles outside $\overline{D_1(0)}$ (including all polys, which have residue free poles at the pt at ∞).

Def^n Bergman span at a pt $a \in \Omega = \mathcal{K}_a =$ finite linear combos of $K_a^{(m)}(z)$ for $m = 0, 1, 2, \ldots$

Fact: $\mathcal{K}_0$ for unit disc $=$ all polys.

<u>Consequence</u> Construction to approx a smooth s.c. domain $\neq \mathbb{C}$ by a Q.D. only used Taylor approx to $F = f_a^{-1}$.



"Improved Riemann mapping thm" because $\tilde{\Omega}$ is a one-pt Q.D. that is as close to $\tilde{\Omega}$ as desired!

<u>Thm</u>: S.C. $\Omega \neq \mathbb{C}$ are Q.D. if and only if inverse of Riemann map is a rational fcn.

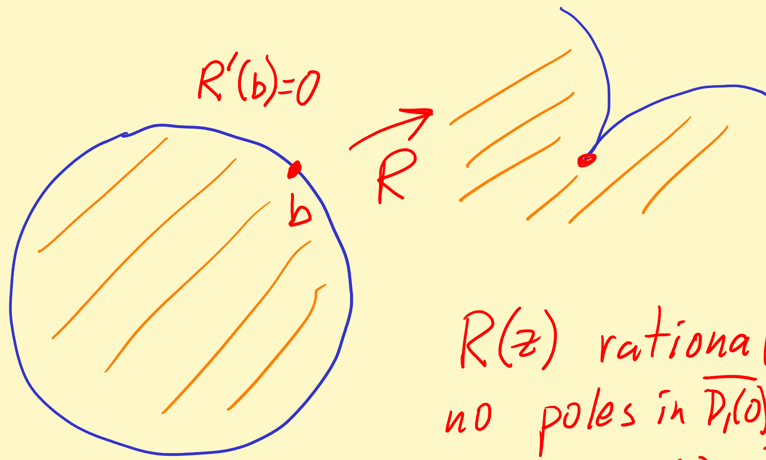
<u>Cor</u>: If $R(z)$ is rational, no poles in $\overline{D_1(0)}$, and 1-1 on $D_1(0)$, then $R(D_1(0))$ is a Q.D.

<u>Cor</u>: Boundary of a s.c. Q.D. is a "real analytic

curve" $R(C_1(0))$, R rational.

Hmmm. Most s.c. Q.D. have C^∞-smooth real analytic bndries, but ...

this is possible:



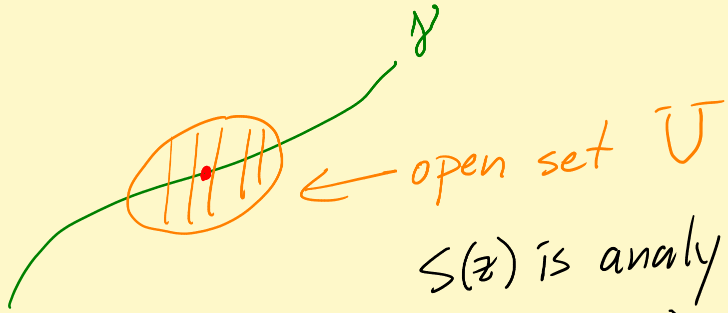
$R(z)$ rational, no poles in $\overline{D_1(0)}$, 1-1 in $D_1(0)$.

R' non-vanishing inside, $R'(b)=0$

Q.D. can have finitely many cusps in bndry, but these are exceptional and can be avoided in approx problems.

Remark C^∞ boundary behavior of Riemann map in smooth setting allowed us to get $\tilde{\Omega}$ C^∞-close to Ω. Carathéodory thm allows us to get $\tilde{\Omega}$ just uniformly close to Ω.

The Schwarz fcn of a smooth real analytic curve.



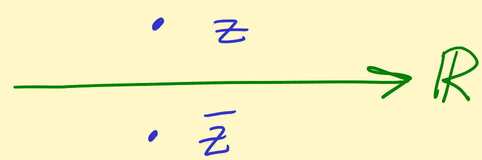
$S(z)$ is analytic on U
and $S(z) = \bar{z}$ on γ.

<u>Fact</u>: Given pt $a \in \gamma$, $\exists$ such a U and S $(a \in U)$.

<u>Note</u>: If we have S_1 and S_2 on U_1 and U_2, then $S_1(z) - S_2(z) = \bar{z} - \bar{z} \equiv 0$ on γ near a.

So $S_1 \equiv S_2$ on $U_1 \cap U_2$. S is "locally unique".

<u>Classic examples</u> Real line



$z \mapsto \bar{z}$ is the "Schwarz reflection fcn"

$R(z) = \bar{z}$ is antiholomorphic, 1-1,

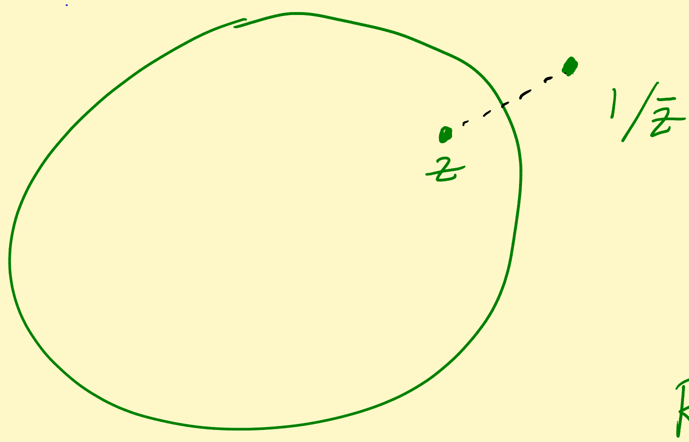
sends above $\mathbb{R}$ to below $\mathbb{R}$,

$R(z) = z$ on $\mathbb{R}$, <u>and</u> $R \circ R = Id$.

Schwarz reflection fcn is an antiholomorphic reflection fcn.

Schwarz fcn $S(z) = \overline{R(\bar{z})}$ is holo.

EX:



$z = re^{i\theta}$

$\bar{z} = re^{-i\theta}$

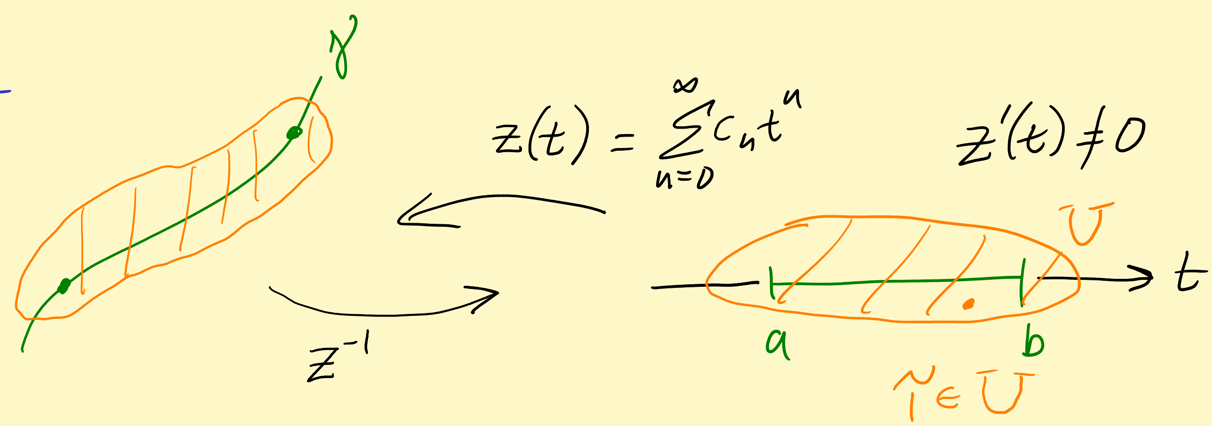
$\frac{1}{\bar{z}} = \frac{1}{r}e^{i\theta}$

$R(z) = 1/\bar{z}$

$S(z) = 1/z$

$S(e^{i\theta}) = e^{-i\theta} = \overline{e^{i\theta}}$ ✓

In general



$z(t) = \sum_{n=0}^{\infty} c_n t^n \qquad z'(t) \neq 0$

$z(\tilde{\tau})$ is analytic with an analytic inverse.

$R(\zeta) = z\left(\overline{z^{-1}(\zeta)}\right) =$ anti-holo reflection

Schwarz fcn $\quad S(\zeta) = \overline{R(\zeta)} = \overline{z\left(\overline{z^{-1}(\zeta)}\right)}$.

Closed real analytic curve: Get analytic $S(z)$ on a "collared" nbhd of curve.