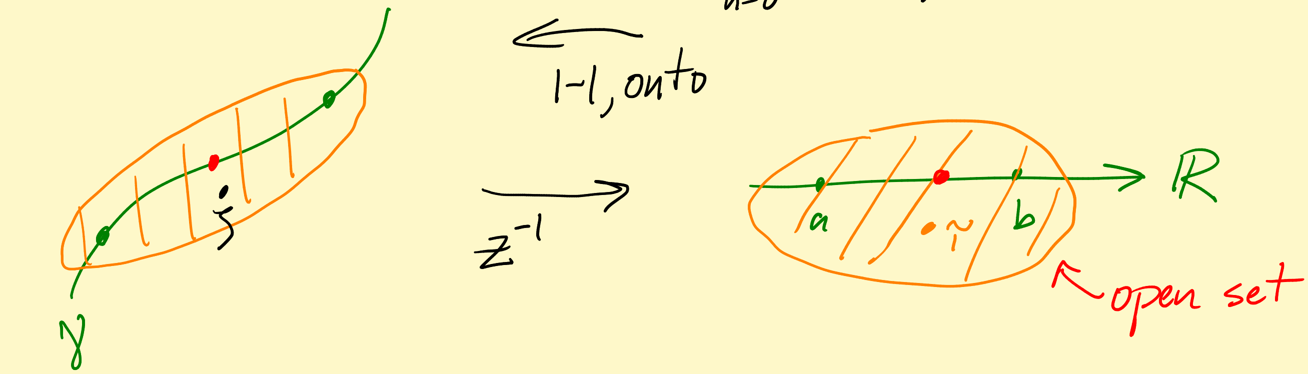
Lecture 41 Schwarz fcn & algebraic kernel fcns

γ real analytic curve

$$z(t) = \sum_{n=0}^{\infty} c_n t^n, \quad z'(t) \neq 0$$



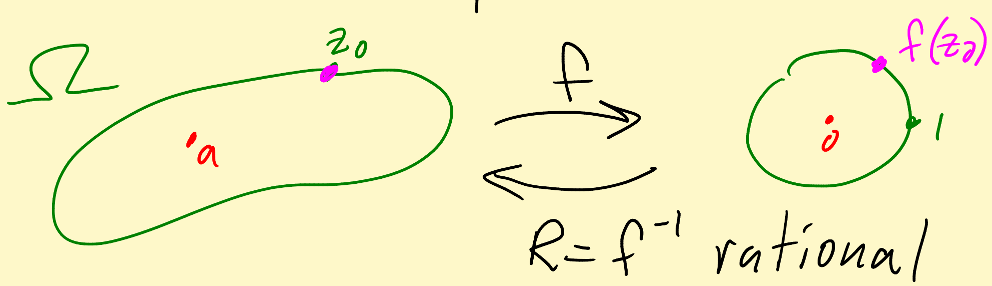
Reflection: $z\left(\overline{z^{-1}(\zeta)}\right)$ ← reflection

Schwarz fcn: $\overline{z\left(\overline{z^{-1}(\zeta)}\right)} = \underbrace{S(\zeta)}_{\text{analytic near } \gamma} = \bar{\zeta}$ on γ.

Hmmm. Use $C_1(0)$ in place of $\mathbb{R}$.

($S(z)$ for $C_1(0)$ is $S(z) = 1/z$, reflection $= 1/\bar{z}$)

Thinking: S.C. quad domain $\Omega \neq \mathbb{C}$ is a rational image of unit disc: inverse of a Riemann map is rational.



Reflection for $b\Omega$: $\quad R\Big(\underbrace{\text{Reflect}(f(z))}_{1/\overline{f(z)}}\Big)$

Schwarz fcn: $\quad \overline{R\left(1/\overline{f(z)}\right)}$

$\underline{\text{analytic near } b\Omega}$ ✓

$\overline{R(\underbrace{1/\overline{f(z_0)}}_{f(z_0)})} = \overline{z_0}$

$S(z_0) = \overline{z_0}$ ✓

Aha! As we move z into Ω from nbhd of $b\Omega$, $1/\overline{f(z)}$ goes outside $\overline{D_1(0)}$. R is well defined outside! Conclude: $S(z)$ extends to Ω as a meromorphic fcn with no poles on $b\Omega$ and <u>finitely</u> many poles in Ω.

<u>Thm</u>: A simply connected domain $\neq \mathbb{C}$ is a quadrature domain $\Longrightarrow$ $b\Omega$ is real analytic, inverse of Riemann map is rational $\Longrightarrow$ $S(z)$ extends mero to Ω.

Thm: Reverse implications are true!

Pf: Suppose γ is a real analytic Jordan curve. Schwarz fcn $S(z)$ is analytic on a collared nbhd of γ. Suppose $S(z)$ extends mero to Ω = inside of γ. Let $h \in H^2(\Omega)$.

$$\iint_\Omega h \, dA = \frac{i}{2} \iint_\Omega h \quad dz \wedge d\bar{z}$$

$$= \frac{i}{2} \iint_\Omega \underbrace{\frac{\partial}{\partial \bar{z}} \left[h(z)\, \bar{z} \right]}_{=h \ \checkmark} \left(- d\bar{z} \wedge dz \right)$$

Green's

$$= -\frac{i}{2} \int_{b\Omega} h(z)\, \bar{z} \; dz$$

$\bar{z} = S(z)$ on $b\Omega$

$$= \frac{-i}{2} \int_{b\Omega} h(z)\, S(z) \; dz$$

$$= \frac{-i}{2} \left[2\pi i \sum_{a_n \in \Omega} \mathrm{Res}_{a_n} hS \right] = q.i.$$

where $\mathcal{O}$ = poles of S in Ω.

Get a q.i. (that does depend on h). ✓

History Harold Shapiro explored connection between Q.D. and Schwarz fcn. His student, Björn Gustaffson, extended ideas to domains with holes and Riemann surfaces. [A domain with holes is a Q.D. $\Leftrightarrow$ z extends meromorphically to double.

$f : \Omega_1 \to \Omega_2$: Ω_2 is a Q.D.

$\Leftrightarrow$ f extends mero to double of Ω_1.

Beautiful geometric theory!]

Fun facts: Ω a s.c. Q.D. $\neq \mathbb{C}$. Then

$$\Omega = R(D_1(0)) \qquad R = f^{-1} \text{ rational.}$$

So $f = R^{-1}$ is algebraic! $\Rightarrow$ f' algebraic

$$K_\Omega(z,w) = \frac{f'(z)\overline{f'(w)}}{\pi(1 - f(z)\overline{f(w)})^2} \quad \text{is algebraic!}$$

$$K_\Omega(z,w) = 4\pi\, S_\Omega(z,w)^2$$

So $S_\Omega(z,w)$ is <u>algebraic</u> too!

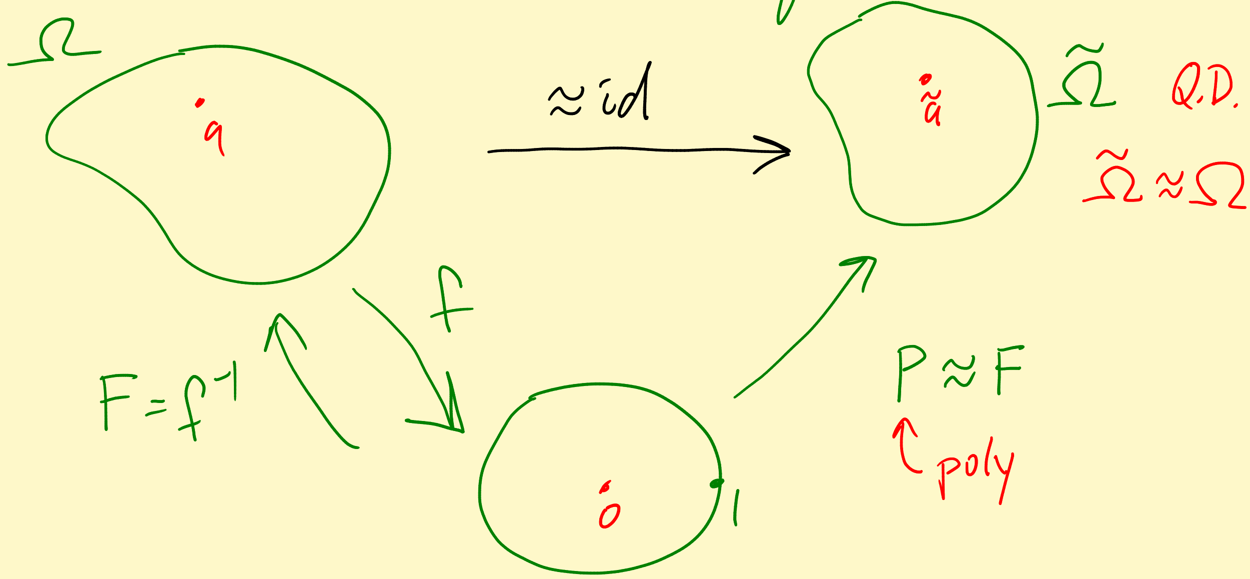
Poisson kernel: $$p(z,w) = \frac{S(z,w)S(w,a)}{S(z,a)} + \overline{\frac{S(z,w)L(w,a)}{L(z,a)}}$$

$S(w,z) = \overline{S(z,w)}$

set $a = z$: $$= \underbrace{\frac{|S(z,w)|^2}{S(z,z)}}_{\text{real algebraic fcn.}}$$

pole of L at $z=a$ makes conjugate term vanish

<u>Moral</u>: Improved Riemann mapping thm



$\tilde\Omega$ is like unit disc: All kernels are explicit algebraic fcns only depending on finitely many coefficients of a polynomial (or rational fcn).