Lecture 42 Arc-length quadrature domains (Chap 23)

Assume Ω is a simply connected domain bounded by a C^∞-smooth Jordan curve. Ω is an arc-length quadrature domain (a-l q.d.) if

$$(*) \quad \int_{b\Omega} h \underbrace{ds}_{\text{arc length}} = \underbrace{\sum_{n=1}^{N} \sum_{m=0}^{M_n} c_{nm} h^{(m)}(a_n)}_{\text{q.i. of order } M = \text{Max } M_n\text{'s over } N\text{-pts.}}$$

for all $h \in H^2(b\Omega)$. ← Hardy space

Note: $\displaystyle\int_{b\Omega} h\, ds = \langle h, 1 \rangle_{H^2(b\Omega)}$

$S(z,w)$ = Szegő kernel for Ω

Lecture 42 a-ℓ Quadrature domains

Ave property ⟹ discs are "quadrature domains" (a-ℓ)

$$h(a) = \frac{1}{\ell(C_r(a))} \int_{C_r(a)} h(z)\, ds$$

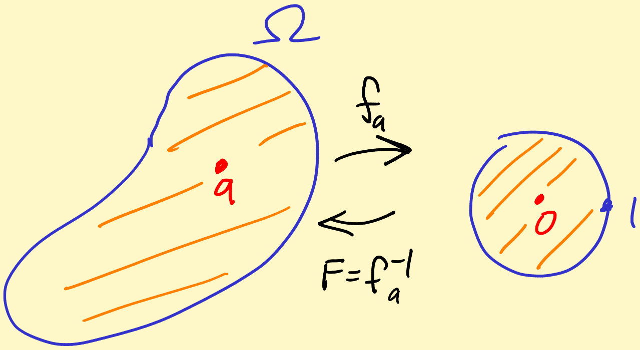
← One point quad. identity of order 0.

$h \in H^2(C_1(a))$

So $h(a) = \langle h, S_a \rangle$ where $S_a(z) = \dfrac{1}{\ell(C_r(a))}$

Fact (Epstein) Discs are only domains with one point a-ℓ q.i.'s of order 0.

Pf in smooth simply connected case



Suppose $\displaystyle\int_{b\Omega} h\, ds = \langle h, 1 \rangle_{b\Omega} = c\, h(a)$ ← q.i.

for all $h \in H^2(b\Omega)$.

a is a fixed pt and c does not depend on h.

Let $h \equiv 1$ to see $c = \ell(b\Omega)$

Claim $S_a \equiv \dfrac{1}{\ell(b\Omega)}$ ← because it represents point evaluation at a functional

$$S_{D_1(0)}(z,w) = \frac{1}{2\pi(1-z\bar{w})}$$

Transformation formula for Szegő kernels:

$$\sqrt{f_a'(z)}\,\underbrace{S_{D_1(0)}(f_a(z),0)}_{1/2\pi\cdot 1} = \underbrace{S(z,\underbrace{F(0)}_{a})}_{S_a(z)=\frac{1}{\ell(b\Omega)}}\overline{\sqrt{\underbrace{F'(0)}_{\text{constant}}}}$$

Aha! $\sqrt{f_a'(z)} \equiv \text{const} = A \implies f'(z) \equiv \text{const}$

So $f_a(z) = Az + B$

So F is linear too, and $\Omega =$ disc. ✓

<u>Fun thing</u>: Look for Ω that are two-point q.d. of order 0. S.C. case: Get Riemann map mapping one pt to $0 \in D_1(0)$. Play around with transf formula.

Simply connected case.

$$S(z,w) = \sqrt{f'(z)}\,S_{D_1(0)}(f(z),f(w))\overline{\sqrt{f'(w)}}$$

$$= \frac{f'(z)\overline{f'(w)}}{2\pi(1-f(z)\overline{f(w)})} \leftarrow \text{nice!}$$

see $S(z,w)$ analytic in z, anti-holo in w.

$$h(z) = \int_{b\Omega} S(z,w)\,h(w)\,ds_w = \langle h, S_z\rangle \leftarrow \text{diff w.r.t. } z\text{. let } z=a$$

Notation $$S_a^{(m)}(z) = \frac{\partial^m}{\partial \bar{w}^m} S(z,w)\Big|_{w=a}$$

convention: $S_a^{(0)}(z) = S_a(z) = S(z,a)$.

$$h(a) = \langle h, S_a \rangle = \iint_\Omega h(z)\, \overline{S(z,a)}\, dA$$

$$h^{(m)}(a) = \langle h, S_a^{(m)} \rangle \leftarrow$$ differentiate in a under $\iint$ and use formula for $S(z,w)$.

$S_a^{(m)}$ represents fcn^l $h \mapsto h^{(m)}(a)$

Easy fact Ω satisfies a-l q.i. $(*)$ $\iff$

$$1 \equiv \underbrace{\sum_{n=1}^{N} \sum_{m=0}^{M_n} \overline{c_{nm}}\, S_{a_n}^{(m)}}_{\text{element of "Szegő span"}}$$

Pf $$\int_{b\Omega} h\, ds = \langle h, 1 \rangle_{b\Omega}$$

Claim $$= \langle h, s \rangle$$

$$= \sum_{n=1}^{N} \sum_{m=0}^{M_n} \langle h, \overline{c_{nm}}\, S_{a_n}^{(m)} \rangle$$

$$= \sum_{n=1}^{N} \sum_{m=0}^{M_n} c_{nm}\, h^{(m)}(a_n) \leftarrow \text{q.i.} \checkmark$$

So $\langle h, \underbrace{1 - \mathcal{S}} \rangle = 0$ for $\underline{\text{all}}$ $h \in H^2(b\Omega)$.

$\in H^2(b\Omega)$ $\perp$ to $H^2(b\Omega)$ $\Rightarrow$ $1 - \mathcal{S} \equiv 0$.

<u>Thm</u>: Suppose $f : \Omega_1 \to \Omega_2$ is a biholomorphic map between domains. Ω_2 is a-l quadrature domain $\iff$ $\sqrt{f'}$ belongs to the "Szegő span" of Ω_1

(= complex linear span $\mathcal{S}_1$ of

$$\left\{ S_{1\,a}^{(m)} \; ; \; a \in \Omega, \; m = 0, 1, 2, \ldots \right\})$$

<u>Pf</u>:

$$\int_{b\Omega_2} h \, ds = \iint_{\Omega_1} \underbrace{|f'|}_{} (h \circ f) \, ds$$

dilation of arc-length in change of var. formula for curves

$$= \iint_{\Omega_1} \sqrt{f'} (h \circ f) \underbrace{\overline{\sqrt{f'}}}_{} \, ds$$

Assume $\sqrt{f'} = \sum c_{mn} S_{1\,a_n}^{(m)}$

$$= \sum \overline{c_{nm}} \left\langle \sqrt{f'} (h \circ f), S_{1\,a_n}^{(m)} \right\rangle$$

$$= \sum \overline{c_{nm}} \; \frac{\partial^m}{\partial z^m} \left[\sqrt{f'}\,(h \circ f) \right] \Bigg|_{z = a_n}$$

= q.i. = finite linear combo of values and derivatives of h at points in $\{ f(a_n) : n = 1, \ldots, N \}$

Proof of ($\Leftarrow$) is done. Easy.

Proof of ($\Rightarrow$): Hmmm. $\langle h, \sqrt{f'} \rangle_{b\Omega_1} =$

$$\langle h, \underbrace{\sqrt{f'}\,(1 \circ f)}_{\lambda_1(1)} \rangle_{b\Omega_1} = \langle \lambda_2 h, 1 \rangle_{b\Omega_2}$$

where $\lambda_2 h = \sqrt{F'}\,(h \circ F)$ where $F = f^{-1}$.

Aha! $\cdots = \displaystyle\int_{b\Omega_2} \sqrt{F'}\,(h \circ F)\, ds$.

If Ω_2 is a q.d. $\cdots =$ lin combo of values and derivatives at $\{a_n\}_{n=1}^N$ in Ω_2.

= lin combo of values and deriv of h at $\{F(a_n)\}_{n=}^N$ in Ω.

$= \langle h, s \rangle$ for the appropriately

written $s \in \mathcal{S}_1$.

[see pf that $\overset{a\text{-}\ell}{q.i.}$ equiv to $1 \in \mathcal{S}$.]

$$\langle h, \sqrt{f'} \rangle_{b\Omega_1} = \langle h, s \rangle_{b\Omega_1} \text{ for all } h \in H^2(\Omega_1).$$

$\Longrightarrow \sqrt{f'} = s$. So $\sqrt{f'} \in \mathcal{S}_1$. ✓

Important application If $\Omega \neq \mathbb{C}$ and simply connected, Riemann map $f : \Omega \to D_1(0)$, and Thm $\Longrightarrow$ $\left[\Omega \text{ is a } \overset{a\text{-}\ell}{q.d.} \iff \sqrt{F'} \in \mathcal{S}_{D_1(0)}\right]$

where $F = f^{-1}$.

$\mathcal{S}_{D_1(0)}$:

$$S(z,w) = \frac{1}{\pi(1 - z\bar{w})} = \frac{1}{-\bar{w}\pi(z - 1/\bar{w})}$$

$$S^{(1)}(z,w) = \frac{z}{\pi(1 - z\bar{w})^2}$$

$$\vdots$$

Szegő span = All rational fcns with poles outside $\overline{D_1(0)}$ (including all polys)

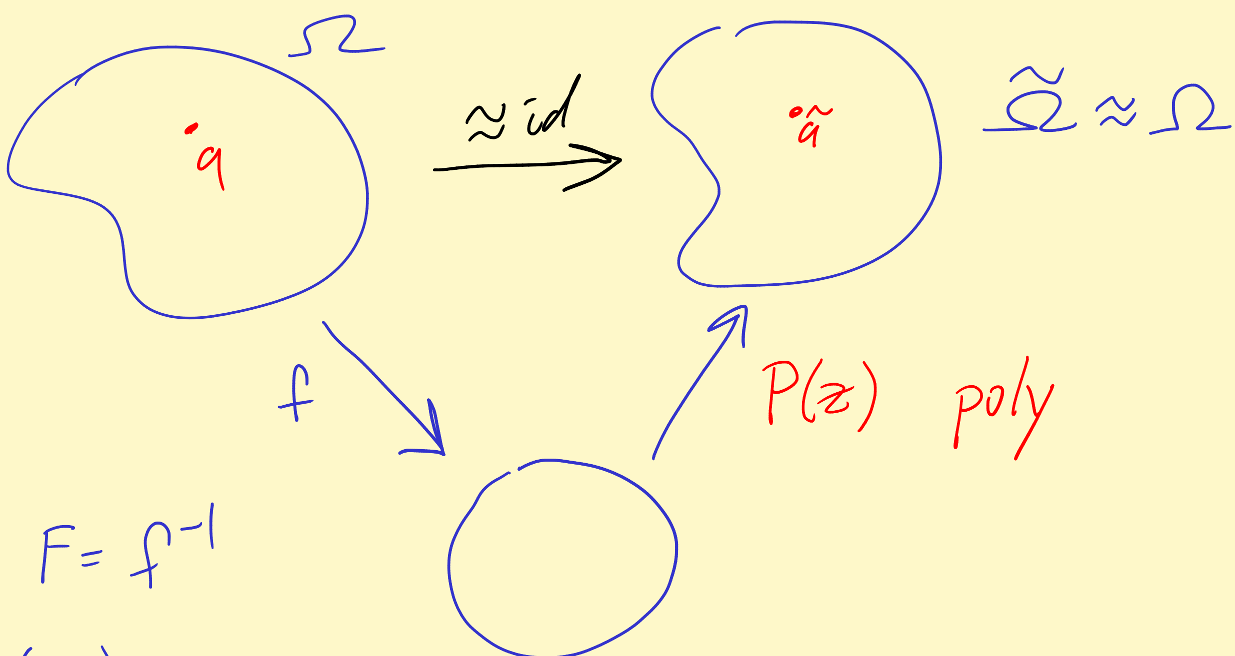
Thm: A smooth bounded simply connected domain is an a-l q.d. $\iff$

$\sqrt{F'}$ is a rational fcn poles outside $\overline{D_1(0)}$

F' = square of such a thing

where F = inverse of a Riemann map.

Remark: Can approx Ω (s.c.) by a-l q.d.



$F(rz)$

$\sqrt{F'(rz)} \approx$ poly $p(z)$ in a nbhd of $\overline{D_1(0)}$

$F' \approx p(z)^2$

$F \approx$ Antiderivative of $p(z)^2$ is a poly $\leftarrow P(z)$

Get $\tilde{\Omega} \approx \Omega$ that is a one-point <u>double</u> q. d. (of order $M \geq 1$).

<u>Other fun things</u> : $\overline{S_a} = \frac{1}{i} L_a T$

a-l q. d. $\overline{\underbrace{\left(\text{Szegő span element}\right)}_{\text{can assume } = 1}} = \sum c_{nm} L_{a_n}^{(m)} T$

Aha! $T = \dfrac{1}{\text{Garabedian span element}}$

... extends mero to Ω!

$\overline{T} = 1/T$ does too!

If and only if :

$$\int_{b\Omega} h \, ds = \int_{b\Omega} h \, \overline{T} \cdot \underbrace{T \, ds}_{dz}$$

$\overline{T} = G(z)$ mero on Ω

$$= \int_{b\Omega} h \, G \, dz$$

$$= 2\pi i \sum_{\text{poles of } G} \text{Res}\,(hG) \quad = \text{q. i.}$$

What about Schwarz?

$$S(z) = \bar{z}$$

parametrize w.r.t. arc-length s

$$S(z(s)) = \bar{z}(s) \quad \text{on} \quad b\Omega$$

Take $\frac{d}{ds}$: $\quad S'(z(s))\, z'(s) = \overline{z'(s)}$

$$S'(z)\, T(z) = \overline{T(z)} \quad \text{on } b\Omega$$

$$S'(z) = \overline{T(z)}^{\,2} \quad \text{extends mero to } \Omega.$$

Many results extend to domains with holes.