Lecture 43 Hartogs' (easy) theorem

Can solve $\bar\partial$ in compact support category in $\mathbb{C}^n$, $n>1$.

Review in $\mathbb{C}$:
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial u}{\partial z}\underbrace{dz}_{dx+i\,dy} + \frac{\partial u}{\partial \bar z}\underbrace{d\bar z}_{dx-i\,dy}$$

$$d = \partial + \bar\partial : \quad \partial u = \frac{\partial u}{\partial z}dz \qquad \bar\partial u = \frac{\partial u}{\partial \bar z}d\bar z$$

Easy facts: $d^2=0 \leftarrow$ mixed partials are equal

$\partial^2=0$, $\bar\partial^2=0 \leftarrow$ same

Recall: Solved $\frac{\partial u}{\partial \bar z} = v$ given $v\in C_0^\infty(\mathbb{C})$.

Sol^n
$$u(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{v(w)}{w-z}\,dw\wedge d\bar w$$

Tool:
$$u(z) = \frac{1}{2\pi i}\int_{C_R(0)} \frac{u(w)}{w-z}\,dw + \frac{1}{2\pi i}\iint_{D_R(0)} \frac{\frac{\partial u}{\partial \bar w}(w)}{w-z}\,dw\wedge d\bar w$$

Note: If $\varphi\in C_0^\infty(\mathbb{C})$, then $\quad \varphi\in C_0^\infty(D_R(0))$

$$\iint_{\mathbb{C}} \frac{\partial\varphi}{\partial \bar z}\,dA = 0$$

$$= \iint_{\mathbb{C}} \frac{\partial\varphi}{\partial \bar z}\left(-\frac{i}{2}\,d\bar z\wedge dz\right) \underset{\text{Green's}}{=} \frac{-i}{2}\int_{C_R(0)} \varphi\,dz = 0$$

Simple consequence: Not always possible, given $V \in C_0^\infty(\mathbb{C})$, to find $u \in C_0^\infty(\mathbb{C})$ with $\frac{\partial u}{\partial \bar{z}} = V$.

__Why__: e.g., pick $V =$ bump fcn of positive mass so that $\iint_{\mathbb{C}} v \, dA \neq 0$.

__Way different in $\mathbb{C}^n$__, $n>1$.

__Hartogs' thm__: Given a $(0,1)$ form

$$V = \sum_{j=1}^{n} V_j \, d\bar{z}_j \quad \text{where} \quad V_j \in C_0^\infty(\mathbb{C}^n)$$

such that $\frac{\partial V_i}{\partial \bar{z}_j} = \frac{\partial V_j}{\partial \bar{z}_i}$ for i, j, $\leftarrow$ compat. cond.

there exists $u \in C_0^\infty(\mathbb{C}^n)$ such that

$$\bar{\partial} u = V, \quad \text{i.e.,} \quad \underbrace{\bar{\partial} u = \sum_{j=1}^{n} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j = V}_{\frac{\partial u}{\partial \bar{z}_j} = V_j \; : \; j=1, \dots, n}$$

__Note__: Compat. cond.: If $\bar{\partial} u = V$, then

$$\underbrace{\bar{\partial}\bar{\partial} u}_{0} = \bar{\partial} V.$$

So $\bar{\partial} V = 0$

$$\sum \left(\frac{\partial V_i}{\partial \bar{z}_j} - \frac{\partial V_j}{\partial \bar{z}_i} \right) d\bar{z}_j \wedge d\bar{z}_i$$

Easy way: $\frac{\partial u}{\partial \bar{z}_j} = v_j \leftarrow$ take $\frac{\partial}{\partial \bar{z}_i}$

$$\frac{\partial^2 u}{\partial \bar{z}_i \partial \bar{z}_j} = \frac{\partial v_j}{\partial \bar{z}_i}.$$

Repeat $\frac{\partial u}{\partial \bar{z}_i} = v_i \leftarrow$ take $\frac{\partial}{\partial \bar{z}_j}$, etc.

<u>Pf</u>: Amazing luck!

$$u(z_1, z_2, \dots, z_n) = \frac{1}{2\pi i} \iint_{\zeta_1 \in \mathbb{C}} \frac{v_1(\zeta_1, z_2, z_3, \dots, z_n)}{\zeta_1 - z_1} \, d\zeta_1 \wedge d\bar{\zeta}_1$$

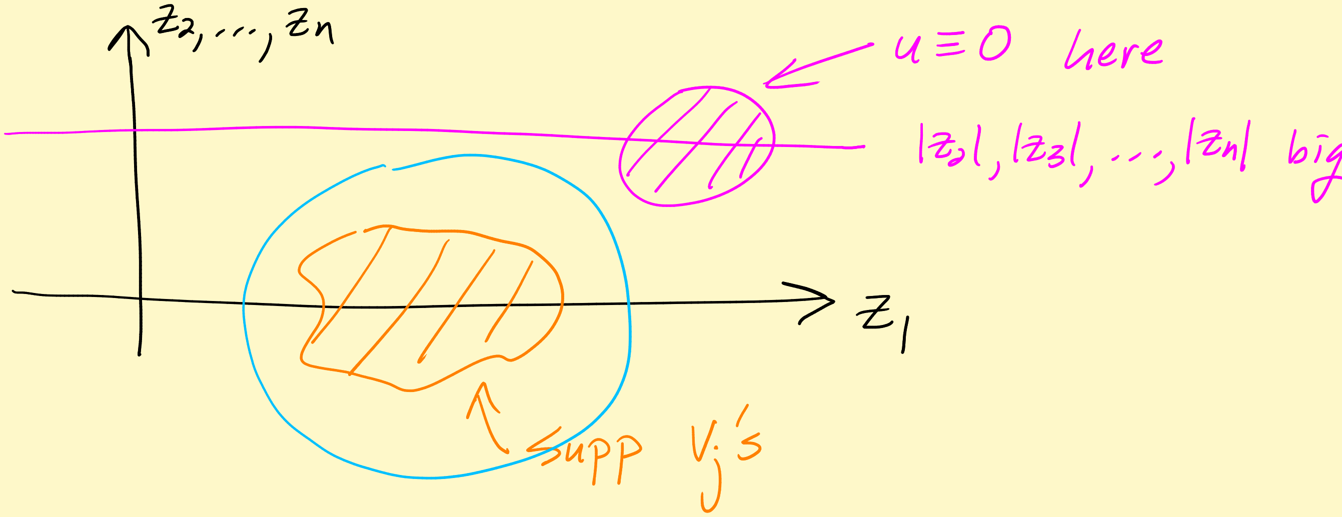
Know $\frac{\partial u}{\partial \bar{z}_1} = v_1$.

Diff under $\iint$:

$j \geq 2$
$$\frac{\partial u}{\partial \bar{z}_j} = \frac{1}{2\pi i} \iint_{\zeta_1 \in \mathbb{C}} \frac{\overbrace{\frac{\partial v_1}{\partial \bar{z}_j}}^{= \frac{\partial v_j}{\partial \bar{\zeta}_1}}(\zeta_1, z_2, \dots, z_n)}{\zeta_1 - z_1} \, d\zeta_1 \wedge d\bar{\zeta}_1$$

Aha! Use Improved Cauchy integral on a big $D_R(0)$. See $\frac{\partial u}{\partial \bar{z}_j} = v_j$ ✓

<u>Last step</u>: Show u has compact support. Easy!



Note: $\bar{\partial} u = v$. So u holo outside supp v's

Uniqueness thm in $\mathbb{C}^n$ $\Rightarrow$ $u \equiv 0$ outside blue ball.

So $u \in C_0^\infty(\mathbb{C}^n)$.

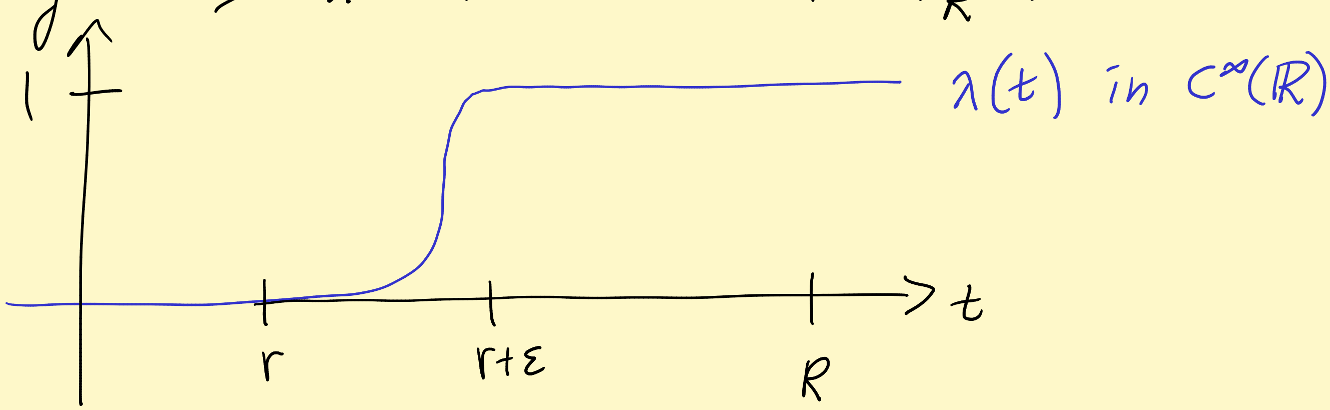
See p. 30 of Hörmander, Chap 2.

Consequence: No isolated singularities in $\mathbb{C}^n$, $n > 1$!

Why: Suppose h holo on $B_R(0) - \overline{B_r(0)}$. $R > r > 0$.

Hartogs $\Rightarrow$ h extends holo to $B_R(0)$.

Pf:



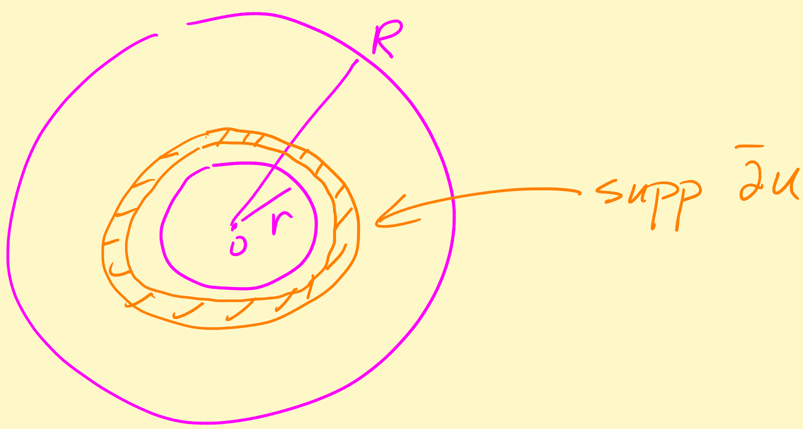
$$\chi(z) = \lambda(|z|) \quad \text{where } |z|^2 = \sum_{j=1}^{n} |z_j|^2$$

Define

$$u = \begin{cases} \chi h & B_R - \overline{B_r} \\ 0 & B_r \end{cases}$$

Notice that

$$\bar\partial u = \begin{cases} h\,\bar\partial\chi & B_R - \overline{B_r} \\ 0 & B_r \end{cases}$$



Let

$$v = \begin{cases} 0 & \text{outside } \overline{B_R} \\ h\,\bar\partial\chi & \text{in } B_R - \overline{B_r} \\ 0 & \text{in } B_r \end{cases}.$$

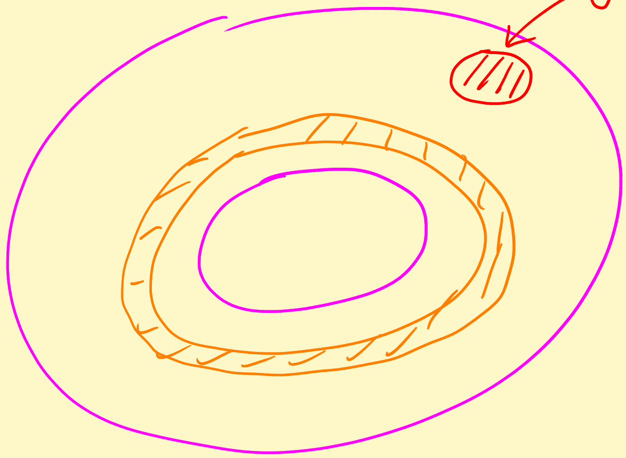
$v \in C_0^\infty(\mathbb{C}^n)$.

Aha! Since $v = \bar\partial(\chi h)$ on its support, it satisfies compat. cond.

So Hartogs yields $U \in C_0^\infty(\mathbb{C}^n)$ with $\bar\partial U = v$.

<u>Claim</u>: $u - \bar{U}$ is holo fcn on B_R that agrees with h on $B_R - \bar{B}_r$.

$\bar{U} \equiv 0$ here. So $\bar{U} \equiv 0$ on $|z| > r + \varepsilon$



$\bar{\partial}$

Check: $\bar{\partial}(u - \bar{U}) = \bar{\partial} u - v = 0$ ✓ $u - \bar{U}$ holo on B_R

See $u - \bar{U} = h$ on $\{ r + \varepsilon < |z| < R \}$.

And since $u - \bar{U}$ is holo, $= h$ on $\{ r < |z| < R \}$ by Uniqueness thm.