MA 693 Lecture 39

Wednesday, April 21, 2021 1:22 PM

$$G(z,w) = -\ln|z-w| + u_w(z)$$

$$\begin{cases} u_w(z) \ \text{harmonic} \\ u_w(z) = \ln|z-w| \ \text{when } z \in b\Omega \end{cases}$$

$$(*) \quad \frac{\partial G}{\partial z}(z,w) = \underbrace{\frac{-1}{2(z-w)}}_{} + \frac{\partial u_w(z)}{\partial z}$$

Notice that $\underbrace{G(w,z)}_{=0,\ w \in b\Omega} = G(z,w)$

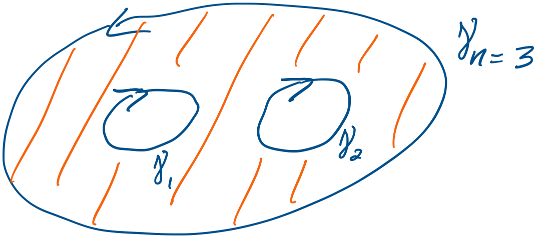
So $\frac{\partial G}{\partial z}(z,w) \equiv 0$ when $z \in \Omega$, $w \in b\Omega$.

1) $\underbrace{\frac{\partial G}{\partial z}(z,w) + \frac{1}{2(z-w)}}_{v}$ is harmonic in w. ✓

2) $v(w) = \frac{1}{2(z-w)}$ when $w \in b\Omega$.

$$K_a = \frac{1}{\pi}\frac{\partial}{\partial \bar{z}}\mathcal{E}\left(\frac{1}{z-a}\right)$$

Multiply connected domains



s.c. case: $K(z,w) = 4\pi S(z,w)^2$

$$f_a(z) = \frac{S(z,a)}{L(z,a)}$$

$$f_a'(z) = c\,K(z,a)$$

Key lemma that makes s.c. so nice:

Lemma: If $u \in L^2(b\Omega)$ is $\perp$ to $H^2(b\Omega)$ and $\overline{H^2(b\Omega)}$, then $u \equiv 0$. [Ω is s.c. here. Lemma is false when there are holes.]

Pf: $u = \underbrace{\overline{H}\,\overline{T} = hT}_{\text{only possible if } H = h \equiv 0.}$, $h, H \in H^2(b\Omega)$

Assume $g \in A^\infty(\Omega)$.

$$0 = \int_{b\Omega} g\, \overline{\overline{hT}}\, ds = \int_{b\Omega} g h\, dz \underset{\substack{\uparrow \\ \text{Cauchy's} \\ \text{Thm}}}{=} 0$$

$$= \int_{b\Omega} g\, \overline{H}\,\overline{T}\, ds$$

$$= \int_{b\Omega} g\, \overline{H}\, d\bar{z}$$

$$\underset{\substack{\uparrow \\ \text{Green's}}}{=} \iint_{\Omega} \frac{\partial}{\partial z}\left[g \overline{H} \right] dz \wedge d\bar{z}$$

$$= \iint_{\Omega} g' \, \overline{H} \, \underbrace{dz \wedge dz}_{-2i \, dA}$$

Hmmm. $H \perp \underbrace{\{ g' : g \in A^{\infty}(\Omega) \}}_{\substack{\Omega \text{ s.c. } \Rightarrow \\ \text{this set } = A^{\infty}(\Omega)}}$

But $A^{\infty}(\Omega)$ is dense in Bergman space. So $H \equiv 0$.

Harmonic measure functions:

w_j is a sol^n to D-Prob:

1) w_j harmonic on Ω
2) $w_j \equiv 1$ on γ_j
3) $w_j \equiv 0$ on γ_k, $k \neq j$.

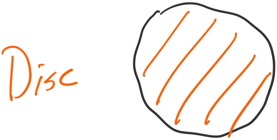
Easy lemma: $\sum_{j=1}^{n} w_j \equiv 1$

Pf: Max, Min princ.

So $w_n = 1 - \sum_{j=1}^{n-1} w_j$.

Analytic fcns related to the w_j's:

Hmmm. Try to cook up harmonic conjugate v_j for w_j.



$$F_j = w_j + i v_j$$

$D \subset \Omega$

Ouch! F_j might not (will not!) be single valued on Ω.

But F_j' is a well-defined analytic fcn on Ω!

$$F_j' = \begin{cases} \frac{\partial}{\partial x} w_j + i \frac{\partial}{\partial x} v_j \\ \underbrace{\frac{\partial v_j}{\partial y} - i \frac{\partial w_j}{\partial y}}_{\text{C-R Eqns}} \end{cases} = \underbrace{\frac{\partial w_j}{\partial x} - i \frac{\partial w_j}{\partial y}}_{2 \frac{\partial w_j}{\partial z}}$$

Defn: $F_j' = 2 \frac{\partial w_j}{\partial z}$ holo on Ω.

Note: 1) F_j' is not the derivative of a global analytic fcn on Ω.

2) $\frac{\partial}{\partial \bar{z}} \left[F_j' \right] = 2 \underbrace{\frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}}_{\frac{1}{4}\Delta} w_j = 0$

So F_j' is holo.

Simple fact:

$$F_n' = -\sum_{j=1}^{n-1} F_j'$$

Harder fact:

$\tilde{\mathcal{F}}'$ = complex linear combos of F_j' : $j = 1, \dots, n-1$

is an $n-1$ dim$^{\ell}$ vector space.

Thm: If $u \perp H^2(b\Omega)$ and $u \perp \overline{H^2(b\Omega)}$, then

$$u = \sum_{j=1}^{n-1} c_j \underbrace{F_j' T}_{-\overline{F_j'}\,\overline{T}} \qquad \text{on } b\Omega$$

Why F_j's come up: Can't always find a single valued antiderivative on domains with holes, but given $g \in \mathcal{O}(\Omega)$, $\exists$ $G \in \mathcal{O}(\Omega)$ with

$$g = G' + \sum_{j=1}^{n-1} c_j F_j'$$

[c_j's uniquely determined].

Multiply connected Ω theorems

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \lambda_{ij} F_i'(z)\overline{F_j'(w)}$$

$$f_a(z) = \frac{S(z,a)}{L(z,a)} \longleftarrow L(z,a) \neq 0 \text{ if } z \neq a.$$

is the Ahlfors map.

$$f_a : \Omega \longrightarrow D_1(0)$$

n-to-one branched cover

$$f_a : \gamma_j \xrightarrow[\text{onto}]{1\text{-}1} C_1(0)$$