MA 693 Lecture 41

Monday, April 26, 2021 1:23 PM

D. Prob: Ω s.c. C^∞-smooth, bndd.

$\perp$ decomp: $\underset{L^2}{\overset{\uparrow}{u}} = h + \bar H \bar T$ $h, H \in H^2(b\Omega)$

$$\left(\frac{T(z)}{2\pi i(z-a)}\right) = h + \bar H T$$

$$h = \overline{[\qquad]}\,\bar T$$

Take conjugate:

$$\overline{S_a} = \frac{1}{i} L_a T \quad \text{on } b\Omega$$

$$= -i L_a T$$

$$\boxed{T = \frac{\overline{S_a}}{-iL_a}} \qquad T\bar T = 1$$

$$\boxed{\bar T = \frac{1}{T} = \frac{-iL_a}{\overline{S_a}}}$$

D. Prob: $u = \operatorname{Re} h$, u harm.

$$u = h + \bar h, \quad h \text{ holo}$$

$$= \left[h + \overline{h(a)}\right] + \underbrace{\overline{\left[h - h(a)\right]}}_{\text{vanishes at } a}$$

$$= h + \bar H \quad \text{where } H(a) = 0.$$

$$= h + \bar H \underbrace{T \bar T}_{1}$$

$$T = \frac{\overline{S_a}}{-iL_a} = 1/\bar T = \frac{i\bar L_a}{S_a}$$

$$u = h + \bar H \underbrace{\left[\frac{i\bar L_a}{S_a}\right]}_{T} \bar T$$

Aha! Mult by S_a

$$S_a u = hS_a + \underbrace{i\underbrace{\bar H \bar L_a}_{\substack{\text{zero of } H \text{ at } a\\ \text{cancels pole of } L_a}} \bar T}_{\perp H^2}$$

See $hS_a = P(S_a u)$.

So $\boxed{h = \dfrac{P(S_a u)}{S_a}}$

See sol^n to D.Prob is in $C^\infty(\bar\Omega)$ when $u \in C^\infty(b\Omega)$.

Neumann prob: Given $\psi \in C^\infty(b\Omega)$, find harmonic fcn u on Ω such that $\frac{\partial u}{\partial n} = \psi$ on $b\Omega$.

Necessary condition on Ψ:

Gauss: $$\int_{b\Omega} \vec{F}\cdot\hat{n}\,ds = \iint_{\Omega} \mathrm{Div}\,\vec{F}\,dA$$

$$\vec{F} = \nabla\varphi$$

$$\int_{b\Omega} \underbrace{\nabla\varphi\cdot\hat{n}}_{\frac{\partial\varphi}{\partial n}}\,ds = \iint_{\Omega} \underbrace{\mathrm{Div}\,\nabla\varphi}_{\Delta\varphi}\,dA$$

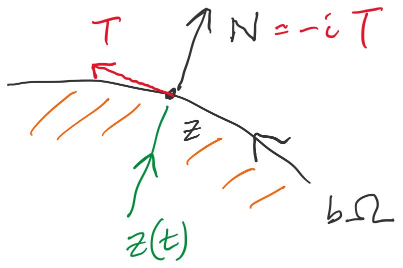
See Ψ must satisfy

$$\boxed{\int_{b\Omega} \Psi\,ds = 0} \quad (*)$$

Hmmm. Say φ solves N. Prob for Ψ satisfying $(*)$.

$$\varphi = h + \bar{h}$$

Normal deriv of h:



$$\frac{d}{dt} h(z(t)) = h'(z(t))\,\underbrace{z'(t)}_{\substack{t = \text{arc length} \\ = N}}$$

$$\frac{\partial}{\partial n} h = h' N = -i h' T$$

$$\varphi = h + \bar{h}$$

$$\frac{\partial\varphi}{\partial n} = -ih'T + \overline{[-ih'T]}$$

$$= -ih'T + i\bar{h}'\bar{T}$$

$$\uparrow$$

$$T = \frac{\bar{S}_a}{-iL_a} = \frac{1}{\bar{T}} = \frac{i\bar{L}_a}{S_a}$$

$$\frac{\partial\varphi}{\partial n} = -i h' \underbrace{\left(\frac{i\bar{L}_a}{S_a}\right)}_{T} + i\bar{h}'\bar{T}$$

Aha! Divide by $\bar{L}_a$ =

$$\frac{1}{\bar{L}_a} \underbrace{\frac{\partial\varphi}{\partial n}}_{\Psi} = \frac{h'}{S_a} + i\bar{h}'\bar{T},$$

get an $\perp$ decomp!

See $$\frac{h'}{S_a} = P\left(\frac{\Psi}{\bar{L}_a}\right)$$

$$\boxed{h' = S_a P(\Psi/\bar{L}_a)}$$

Thm: Given $\Psi \in C^\infty(b\Omega)$ with

$\int_{b\Omega} \psi \, ds = 0$, the sol^n to N. Prop is $\boxed{\varphi = h + \bar{h}}$

where $h' = S_a P(\psi / \bar{L}_a)$.

Note: S_a, L_a are C^∞ smooth up to $b\Omega$ and S_a is non-vanishing on $\bar{\Omega}$ and L_a is non-vanishing on $\bar{\Omega} - \{a\}$. So formula shows that φ is in $C^\infty(\bar{\Omega})$.

Dirichlet to Neumann map:

Start with u on $b\Omega$. Solve D. Prob with bndry data u. Get φ. Take $\frac{\partial \varphi}{\partial n}$ (the Neumann data).

$u \longmapsto \frac{\partial \varphi}{\partial n}$ is D-to-N map.

Interesting observation: D-to-N map takes rational fcns to rational fcns when $\Omega = D_1(0)$.

Why: $\tilde{R}(x,y) \rightsquigarrow R(z, \underset{\uparrow \ \bar{z} = S(z) = \frac{1}{z} \text{ on } C_1(0)}{\bar{z}}) =$

$$R\left(z, \frac{1}{z}\right) \underset{\substack{\uparrow \\ \text{partial} \\ \text{frac}}}{=} \underbrace{\sum \underbrace{\frac{A_{kn}}{(z-a_k)^n}}_{\substack{\text{poles} \\ \text{inside } C_1}}}_{\substack{\text{replace} \\ z \text{ by } 1/\bar{z}}} + \sum \underbrace{\frac{B_{kn}}{(z-b_k)^n}}_{\text{outside}}$$

$\frac{1}{(1 - a_k \bar{z})}$ has pole outside C_1

$$R\left(z, \frac{1}{z}\right) = \overline{r_1(z)} + r_2(z) = \varphi$$

r_1, r_2 rational, poles outside C_1.

φ solves D. prob.

$$\frac{\partial \varphi}{\partial n} = -i\, r_2' \underset{\substack{\uparrow \\ T = iz \text{ on } C_1(0).}}{T} + i\, \overline{r_1'}\, \overline{T}$$

See $\frac{\partial \varphi}{\partial n}$ is rational too!

Thm: Double quad domains satisfy D-to-N: $\left(\begin{smallmatrix}\text{rational} \\ \text{fcns}\end{smallmatrix}\right) \circlearrowleft$.

Any others?