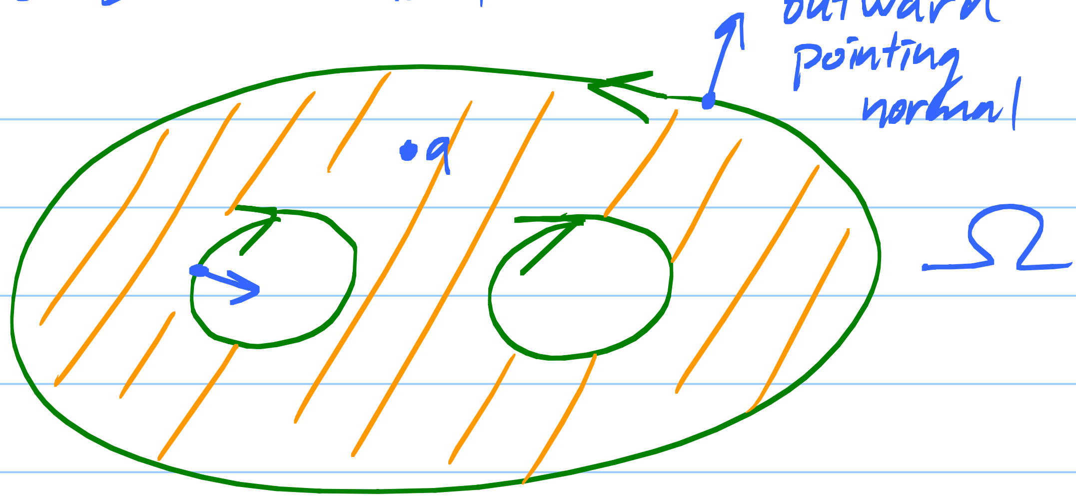
MA 693 Steve Bell



Cauchy Integral Formula

$$h(a) = \frac{1}{2\pi i} \int_{b\Omega} \frac{h(z)}{z-a}\, dz$$

Plan: Cauchy Transform:

u not analytic

$$u \longmapsto (\mathcal{C}u)(a) = \frac{1}{2\pi i} \int_{b\Omega} \frac{u(z)}{z-a}\, dz$$

The whole story:

$$u(a) = \frac{1}{2\pi i} \int_{b\Omega} \frac{u(z)}{z-a}\, dz + \frac{1}{2\pi i} \iint_{\Omega} \frac{\left(\frac{\partial u}{\partial \bar z}\right)}{z-a}\, dz \wedge d\bar z$$

Ingredients in proof: Green's Thm

Differential operators $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar z}$.

Stokes' Thm: $\displaystyle\int_{b\Omega} \alpha = \iint_{\Omega} d\alpha$

Green's Thm: $\alpha = u\,dx + v\,dy$

$$d\alpha = du \wedge dx + dv \wedge dy$$

$$= \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) \wedge dx + \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right) \wedge dy$$

$$= -\frac{\partial u}{\partial y} dx \wedge dy + \frac{\partial v}{\partial x} dx \wedge dy$$

$$\int_{b\Omega} u\,dx + v\,dy = \iint_{\Omega} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx \wedge dy$$

Spivak, *Calculus on Manifolds*.

$b\Omega$ parametrized by $z(t) = x(t) + iy(t)$ $a_j \le t \le b_j$, $j = 1, \dots, n$ $\leftarrow$ n boundary curves.

$$\int_{b\Omega} u\,dx + v\,dy = \sum_{j=1}^{n} \int_{a_j}^{b_j} u(x(t), y(t))\, \dot{x}(t)\,dt + \int_{a_j}^{b_j} v(x(t), y(t))\, \dot{y}(t)\,dt$$

Complex version: $dz = dx + i\,dy$

$d\bar{z} = dx - i\,dy$

$$du = \frac{\partial u}{\partial x}\underset{\frac{1}{2}(dz+d\bar{z})}{\underset{\uparrow}{dx}} + \frac{\partial u}{\partial y}\underset{\frac{1}{2i}(dz-d\bar{z})}{\underset{\uparrow}{dy}} = \Big(\underbrace{\tfrac{1}{2}(u_x - iu_y)}_{\frac{\partial u}{\partial z}}\Big)dz + \Big(\underbrace{\tfrac{1}{2}(u_x + iu_y)}_{\frac{\partial u}{\partial \bar{z}}}\Big)d\bar{z}$$

Def^{n}:
$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$$
$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)$$

Big Fact: u analytic $\iff$ $\dfrac{\partial u}{\partial \bar{z}} = 0$.

If u analytic, then $\dfrac{\partial u}{\partial z} = u'(z)$.

Chain Rule: $u = g(w)$ where $w = f(z)$.

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial u}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial z}$$
$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial \bar{z}} + \frac{\partial u}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial \bar{z}}$$

Conjugates:
$$\overline{\partial u/\partial z} = \partial \bar{u}/\partial \bar{z}$$
$$\overline{\partial u/\partial \bar{z}} = \partial \bar{u}/\partial z$$

Complex Green's Thm:
$$\int_{b\Omega} u\,dz = \iint_{\Omega} \frac{\partial u}{\partial \bar{z}}\,d\bar{z}\wedge dz$$

$$\int_{b\Omega} u\, d\bar{z} = \iint_{\Omega} \frac{\partial u}{\partial z}\, dz \wedge d\bar{z}$$

<u>Facts</u>: $dz \wedge d\bar{z} = (dx + i\,dy) \wedge (dx - i\,dy)$

$= -2i\, dx \wedge dy$

$$\boxed{\begin{aligned} dx \wedge dx &= 0 \\ dy \wedge dy &= 0 \\ dx \wedge dy &= -dy \wedge dx \end{aligned}}$$

$dz \wedge dz = 0$

$d\bar{z} \wedge d\bar{z} = 0$

<u>Pf of Complex Green's</u>:

$$\int_{b\Omega} u\, dz \underset{\text{Stokes}}{=} \iint_{\Omega} d[u\, dz]$$

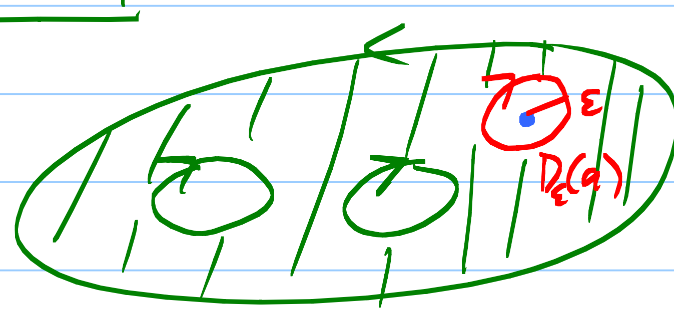
But $d[u\,dz] = du \wedge dz$

$= \left(\frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z}\right) \wedge dz$

$= \frac{\partial u}{\partial \bar{z}}\, d\bar{z} \wedge dz$

<u>Proof of General Cauchy Formula</u>:

Ω_ε



$$\int_{b\Omega_\varepsilon} \frac{u(z)}{z-a}\, dz \underset{\substack{\uparrow \\ \text{Green's}}}{=} \iint_{\Omega_\varepsilon} \frac{\partial}{\partial \bar z}\left(\frac{u(z)}{z-a}\right) d\bar z \wedge dz$$

$$= \iint_{\Omega_\varepsilon} \frac{\left(\frac{\partial u}{\partial \bar z}\right)}{z-a}\, d\bar z \wedge dz$$

Let $\varepsilon \to 0$.

Facts: $\left|\frac{1}{z-a}\, d\bar z \wedge dz\right| = \frac{1}{|z-a|}\left|-2i\, dx \wedge dy\right|$

(polar coords centered at a) $= 2\frac{1}{r}(r\, dr\, d\theta)$

$= 2\, dr\, d\theta$

L^1 singularity! $\iint_{\Omega} \frac{\left(\frac{\partial u}{\partial \bar z}\right)}{z-a}\, d\bar z \wedge dz = \lim_{\varepsilon \to 0} \iint_{\Omega_\varepsilon}$.

$$\int_{b\Omega_\varepsilon} \frac{u}{z-a}\, dz = \int_{b\Omega} \frac{u}{z-a}\, dz - \underbrace{\int_{bD_\varepsilon(a)} \frac{u}{z-a}\, dz}$$

Claim $\longrightarrow 2\pi i\, u(a)$

530: $z(t) = a + \varepsilon e^{it}$ $\quad 0 \le t \le 2\pi$

Collect everything. Use $d\bar{z} \wedge dz = -dz \wedge d\bar{z}$.

Get Cauchy Formula.

The Cauchy Transform, Potential Theory & Conformal Mapping.