$$u(z) = \underbrace{\frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z}\,dw}_{\text{analytic on } \Omega} + \frac{1}{2\pi i}\iint_{\Omega} \frac{\frac{\partial u}{\partial \bar w}}{w-z}\,dw\wedge d\bar w$$

Take $\frac{\partial}{\partial \bar z}$ of formula. See

$$\frac{\partial u}{\partial \bar z} = \frac{\partial}{\partial \bar z}\left(\frac{1}{2\pi i}\iint \frac{\frac{\partial u}{\partial \bar w}}{w-z}\,dw\wedge d\bar w\right)$$

Hmmm. Given $v \in C^\infty(\bar\Omega)$, if we want a u such that $\frac{\partial u}{\partial \bar z} = v$, it seems reasonable to try

$$u(z) = \frac{1}{2\pi i}\iint_{\Omega} \frac{v(w)}{w-z}\,dw\wedge d\bar w.$$

<u>Thm</u>: This solves the $\frac{\partial}{\partial \bar z}$ problem, and if $v \in C^\infty(\bar\Omega)$, then so is u.

<u>Pf</u>: Case: v vanishing to high order s on $b\Omega$.

Then $$u(z) = \frac{1}{2\pi i}\iint_{\Omega} \frac{v(w)}{w-z}\,dw\wedge d\bar w = \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{v(w)}{w-z}\,dw\wedge d\bar w$$

by extending v to be zero on $\mathbb{C}-\bar\Omega$. Note: v is

$C^s(\mathbb{C})$ with compact support in $\overline{\Omega}$.

Change variables: $\zeta = w - z$. $dw \wedge d\bar{w} = d\zeta \wedge d\bar{\zeta}$

Now $u(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{v(\zeta + z)}{\zeta} \, d\zeta \wedge d\bar{\zeta}$.

Since $\frac{1}{\zeta}$ has an L^1 singularity near 0 and v is smooth with compact support,

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial v}{\partial \bar{\zeta}}(\zeta + z)}{\zeta} \, d\zeta \wedge d\bar{\zeta}$$

$$= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial v}{\partial \bar{w}}}{w - z} \, dw \wedge d\bar{w}$$

$$\underset{\text{by Cauchy}}{=} v(z) - \frac{1}{2\pi i} \int_{b\Omega} \frac{\overbrace{v(w)}^{=0 \text{ on } b\Omega}}{w - z} \, dw = v \checkmark$$

Can differentiate under $\iint$ up to order s to see that u is $C^s(\overline{\Omega})$.

<u>General case:</u> $v \in C^\infty(\overline{\Omega})$.

<u>Lemma:</u> Given $v \in C^\infty(\bar\Omega)$ and $s \in \mathbb{N}$, there is a $\varphi \in C^\infty(\bar\Omega)$ such that

1) $\varphi = 0$ on $b\Omega$, and
2) $\frac{\partial \varphi}{\partial \bar z} - v$ vanishes to order s on $b\Omega$.

<u>Pf later</u>. Continue pf of Thm.

$$u(z) - \varphi(z) = \frac{1}{2\pi i}\iint_\Omega \frac{v}{w-z}\,dw\wedge d\bar w - \frac{1}{2\pi i}\iint_\Omega \frac{\frac{\partial\varphi}{\partial\bar w}}{w-z}\,dw\wedge d\bar w$$

by Cauchy formula for φ. (The boundary $\int = 0$).

$$u - \varphi = \underbrace{\frac{1}{2\pi i}\iint \frac{\left(v - \frac{\partial\varphi}{\partial\bar w}\right)}{w-z}\,dw\wedge d\bar w}_{\text{Case 1: } \frac{\partial}{\partial\bar z} \text{ of this } = v - \frac{\partial\varphi}{\partial\bar w}}$$

vanishes to order s on $b\Omega$

Take $\frac{\partial}{\partial\bar z}$ of equation: Get

$$\frac{\partial u}{\partial\bar z} - \frac{\partial\varphi}{\partial\bar z} = \left(v - \frac{\partial\varphi}{\partial\bar z}\right). \quad \text{So } \frac{\partial u}{\partial\bar z} = v,$$

and u is $C^s(\bar\Omega)$ smooth. Take

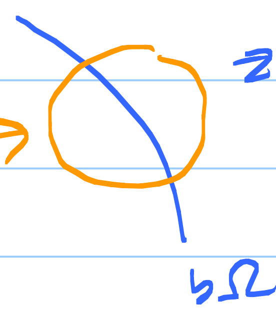
s arbitrarily large from Lemma to see $u \in C^\infty(\bar{\Omega})$ and $\frac{\partial u}{\partial \bar{z}} = v$.

Proof of Lemma: Fact: There is a function ρ in $C^\infty(\bar{\Omega})$ such that

1) $\rho = 0$ on $b\Omega$
2) $\nabla\rho \neq 0$ on $b\Omega$
3) $\Omega = \{\rho < 0\}$.

ρ is a "defining function" for Ω.

How to cook up ρ:



$z(t) = f(t) + i g(t)$

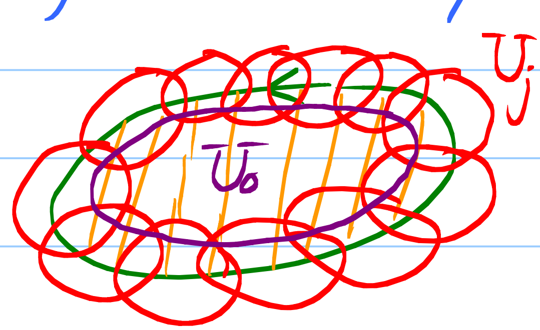
$z'(t) \neq 0$.

Say $f'(t) \neq 0$.

Inverse Fcn Thm: Solve for $t = \sigma(x)$. Eqn for $b\Omega$ $\quad y = g(\sigma(x))$

$$\rho(x,y) = \pm\Big(y - g(\sigma(x))\Big) \text{ works locally.}$$

Finally, use partition of unity:



Have ρ_j on U_j, $j = 1, \ldots, N$.

Let $\rho_0 \equiv -1$ on U_0.

Let $\{\varphi_j\}$ be a C^∞ partition of unity of $\bar{\Omega}$ with respect to cover $\{U_j\}_{j=0}^{N}$.

$$\boxed{\begin{array}{l}\varphi_j \in C_0^\infty(U_j) \\ \sum_0^N \varphi_j = 1 \text{ on } \bar{\Omega}. \\ \varphi_j \geq 0.\end{array}}$$

$$\rho = \sum_{j=0}^{N} \varphi_j \rho_j$$

Only need to check $\nabla\rho = \sum_{j=0}^{N} \varphi_j \nabla\rho_j$ on $b\Omega$. Easy.

How to cook up φ with $\frac{\partial\varphi}{\partial\bar{z}} = v$ to order s on $b\Omega$.

Order 0 case: Try $\varphi = \theta\rho$ where $\theta \in C^\infty(\bar{\Omega})$.

Need $\frac{\partial\varphi}{\partial\bar{z}} = \theta_{\bar{z}}\rho + \theta\rho_{\bar{z}} = v$ on $b\Omega$. want

Hmmm. Try $\theta = \frac{v}{\rho_{\bar{z}}}$.

Note $\rho_{\bar{z}} = \frac{1}{2}\left(\frac{\partial\rho}{\partial x} - i\frac{\partial\rho}{\partial y}\right)$

$\nabla\rho \neq 0$ on $b\Omega$ $\Rightarrow \rho_{\bar{z}} \neq 0$ on $b\Omega$.