Lemma: Given $v \in C^\infty(\bar{\Omega})$, there is a φ in $C^\infty(\bar{\Omega})$ such that $\frac{\partial \varphi}{\partial \bar{z}} - v$ vanishes to order s on $b\Omega$, and such that $\varphi = 0$ on $b\Omega$.

pf continued: Order $s=0$: $\varphi = \theta\rho$ where $\rho \in C^\infty(\bar{\Omega})$,

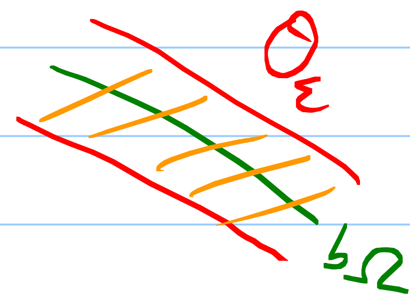
$$\Omega = \{\rho < 0\}$$
$$b\Omega = \{\rho = 0\}$$
$$\nabla\rho \neq 0 \text{ on } b\Omega.$$

Try: $$\frac{\partial}{\partial \bar{z}}(\theta\rho) = \underbrace{\theta_{\bar{z}}\,\rho}_{=0 \text{ on } b\Omega} + \theta\rho_{\bar{z}} \overset{\text{want}}{=} v \text{ on } b\Omega$$

See $\theta = \dfrac{v}{\rho_{\bar{z}}}$ works. Oops!

$$\rho_{\bar{z}} = \frac{1}{2}(\rho_x - i\rho_y) = 0 \text{ where } \nabla\rho = 0.$$

Note: $\rho_{\bar{z}} \neq 0$ on nbhd of $b\Omega$, say $\nabla\rho \neq 0$ on $\mathcal{O}_\varepsilon = \{z : d(z, b\Omega) < \varepsilon\}$



Let μ_δ be a C_0^∞ approx to the identity: $\mu_\delta \in C_0^\infty(D_\delta(0))$,

$\mu_\delta \geq 0$, radially symmetric

$$\iint \mu_\delta \, dx\,dy = 1.$$

Let $\chi = \mu_{\varepsilon/4} * \mathcal{X}_{\varepsilon/2}$ where $\mathcal{X}_{\varepsilon/2} = \begin{cases} 1 & \text{on } \mathcal{O}_{\varepsilon/2} \\ 0 & \text{outside.} \end{cases}$ 2

Notice that $\chi \equiv 1$ on nbhd of $b\Omega$ where $\nabla\rho \neq 0$ and $\chi \equiv 0$ on an open set contianing the compact set where $\rho_{\bar z} = 0$.

Go back: $\varphi = \chi\theta\rho$ where $\theta = \dfrac{v}{\rho_{\bar z}}$.

It works: $\dfrac{\partial\varphi}{\partial\bar z} = \underset{\substack{\uparrow \\ =0 \\ \text{near } b\Omega}}{\chi_{\bar z}}(\theta\rho) + \underset{\substack{\uparrow \\ =1 \\ \text{near } b\Omega}}{\chi}\Big(\theta_{\bar z}\rho + \underbrace{\theta\rho_{\bar z}}_{\substack{=v \\ \text{near } b\Omega}}\Big)$

and $\dfrac{\partial\varphi}{\partial\bar z} - v = \Psi\rho$ where $\Psi \in C^\infty(\bar\Omega)$.

Plan: $\varphi_1 = \theta_1\rho$ as above. $\dfrac{\partial\varphi_1}{\partial\bar z} - v = \Psi_1\rho$

Try $\varphi_2 = \varphi_1 + \theta\rho^2$.

$$\frac{\partial\varphi_2}{\partial\bar z} - v = \underbrace{\frac{\partial\varphi_1}{\partial\bar z} - v}_{\Psi_1\rho} + \theta_{\bar z}\rho^2 + 2\theta\rho\rho_{\bar z}$$

Kill!

choose $\theta = \chi\dfrac{\Psi_1}{2\rho_{\bar z}}$. Get $\dfrac{\partial\varphi_2}{\partial\bar z} - v = \Psi_2\rho^2$.

$\Psi_2 \in C^\infty(\bar\Omega)$.

Induction. Assume we have φ_k such that $\varphi_k = 0$ on $b\Omega$ and $\frac{\partial \varphi_k}{\partial \bar z} - v = \Psi_k \rho^k$ where $\Psi_k \in C^\infty(\bar\Omega)$. Get

$$\varphi_{k+1} = \varphi_k + \theta \rho^{k+1}$$

$$\frac{\partial \varphi_{k+1}}{\partial \bar z} - v = \underbrace{\frac{\partial \varphi_k}{\partial \bar z} - v}_{\Psi_k \rho^k} + \theta_{\bar z} \rho^{k+1} + \theta (k+1) \rho^k \rho_{\bar z}$$

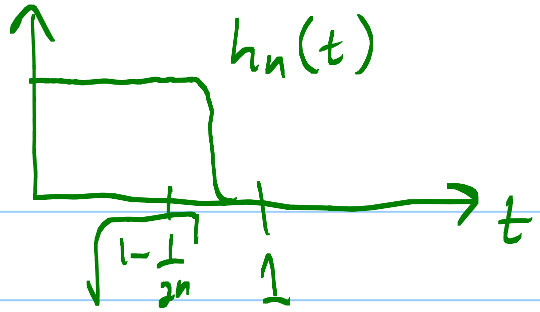
kill!

choose $\theta = \chi \frac{\Psi_k}{(k+1)\rho_{\bar z}}$. Induction complete.

L. Hörmander, Complex Analysis in Several Variables, Chapter 1. ⟸ Math 530 in 17 pages!

<u>Theorem</u>: Let Ω be a domain in $\mathbb{C}$. Given $v \in C^\infty(\Omega)$, there is a $u \in C^\infty(\Omega)$ with $\frac{\partial u}{\partial \bar z} = v$.

<u>Pf in case $\Omega = D_1(0)$.</u> Let χ_n be a cut off function such that $\chi_n \geq 0$, $\chi_n \in C_0^\infty(D_1(0))$ and $\chi_n(z) \equiv 1$ for $|z| < 1 - \frac{1}{2^n}$



$$\chi_n(z) = h_n(|z|^2)$$

Mittag-Leffler Argument:

telescoping

$$\underbrace{\chi_1}_{\sigma_1} + \underbrace{(\chi_2 - \chi_1)}_{\sigma_2} + \underbrace{(\chi_3 - \chi_2)}_{\sigma_3} + \cdots \underbrace{(\chi_n - \chi_{n-1})}_{\sigma_n} = \chi_n$$

Let $u_k \in C^\infty(\overline{D_1(0)})$ solve $\dfrac{\partial u_k}{\partial \bar{z}} = \sigma_k V$.

Notice that $\dfrac{\partial u_k}{\partial \bar{z}} \equiv 0$ on nbhd of $\overline{D_{1-\frac{1}{2^{k-1}}}(0)}$,

meaning u_k analytic there. Get polynomial $p_k(z)$ such that $|u_k - p_k| < \frac{1}{2^k}$.

Now $u = (u_1 - p_1) + (u_2 - p_2) + (u_3 - p_3) + \cdots$

$$= \sum_{n=1}^{N} (u_n - p_n) + \underbrace{\sum_{n=N+1}^{\infty} (u_n - p_n)}_{\substack{\text{converge unif} \\ \text{to analytic fcn} \\ \text{on } D_{1-\frac{1}{2^{N-1}}}(0)}}$$

$$\frac{\partial u}{\partial \bar{z}} = \underbrace{\left(\sum_{n=1}^{N} \sigma_n V\right)}_{\chi_N V \,=\, V \text{ on disc.}} + 0 \quad \text{on disc.}$$

Done!

Hörmander: General Ω, use Runge Thm to get analytic p_n on exhaustion, etc.