Thm: Ω a domain in $\mathbb{C}$. Given $v \in C^\infty(\Omega)$, there is a $u \in C^\infty(\Omega)$ such that $\frac{\partial u}{\partial \bar z} = v$ on Ω.

Mittag-Leffler Thm: Ω domain in $\mathbb{C}$, $\{a_n\}_{n=1}^\infty$ seq of points in Ω with no limit point in Ω. Given principal parts $R_n(z) = \frac{A_{nN}}{(z-a_n)^N} + \cdots + \frac{A_{n1}}{z-a_n}$, there is a meromorphic function f on Ω with poles only at a_n's with principal parts R_n at a_n, meaning $f - R_n$ has a removable sing. at a_n.

Pf: Since $\{a_n\}$ have no limit pt in Ω, there are discs $D_{\varepsilon_n}(a_n)$ such that $\overline{D_{\varepsilon_n}(a_n)} \cap \overline{D_{\varepsilon_m}(a_m)} = \emptyset$ if $n \neq m$ and $\overline{D_{\varepsilon_n}(a_n)} \subset \Omega$ $\forall n$. $\exists\, \chi_n \in C_0^\infty(D_{\varepsilon_n}(a_n))$ such that $\chi_n \equiv 1$ on $D_{\varepsilon_n/2}(a_n)$.

Now $u = \sum_{n=1}^\infty \chi_n R_n$ is a "$C^\infty(\Omega)$ solution" to the M-L problem.

Properties:
1) $u \in C^\infty(\Omega - \{a_n\}_{n=1}^\infty)$
2) $u - R_n$ is analytic$^{=0}$ on $D_{\varepsilon_n/2}(a_n) - \{a_n\}$ with removable sing at a_n.
3) $\frac{\partial}{\partial \bar z} u \equiv 0$ on $D_{\varepsilon_n/2}(a_n) - \{a_n\}$.

Big Idea: Define
$$v = \begin{cases} \frac{\partial u}{\partial \bar z} & \text{on } \Omega - \{a_n\}_1^\infty \\ 0 & \text{at } a_n\text{'s} \end{cases}$$

Notice that v is in $C^\infty(\Omega)$. Now take

$\bar{U}$ in $C^\infty(\bar{\Omega})$ with $\frac{\partial \bar{U}}{\partial \bar{z}} = v$ on Ω.

Now $f = u - \bar{U}$ solves M-L problem!

Hartog's Thm: In $\mathbb{C}^n$,

$$df = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial y_i} dy_i \right)$$

$$= \underbrace{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial z_i} dz_i \right)}_{\partial f} + \underbrace{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i \right)}_{\bar{\partial} f}$$

$\bar{\partial}$-problem. Given $\alpha = \sum \alpha_i \, d\bar{z}_i$, try to find a function f such that $\bar{\partial} f = \alpha$, i.e., such that $\frac{\partial f}{\partial \bar{z}_i} = \alpha_i$, $i = 1, \ldots, n$.

Fact: f is analytic $\Longleftrightarrow$ $\bar{\partial} f = 0$.

Compatibility Conditions:

$$\frac{\partial}{\partial \bar{z}_j} \frac{\partial f}{\partial \bar{z}_i} = \frac{\partial}{\partial \bar{z}_j} \alpha_i$$

$$\|$$

$$\frac{\partial}{\partial \bar{z}_i} \underbrace{\frac{\partial f}{\partial \bar{z}_j}}_{\alpha_j} = \frac{\partial}{\partial \bar{z}_i} \alpha_j$$

To solve $\bar{\partial}$-problem, must have

$$\underbrace{\frac{\partial}{\partial \bar{z}_i} \alpha_j = \frac{\partial}{\partial \bar{z}_j} \alpha_i}_{*} \quad \text{for } i \neq j.$$

Hartog's Thm 1: Given $\alpha_i \in C_0^\infty(\mathbb{C}^n)$ satisfying $(*)$, then $\exists\ f \in C_0^\infty(\mathbb{C}^n)$ with $\bar\partial f = \alpha$ where $\alpha = \sum \alpha_i \, d\bar z_i$. ($n>1$. False if $n=1$.)

Pf: Let
$$u(z_1, z_2, \dots, z_n) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\alpha_1(w, z_2, \dots, z_n)}{w - z_1} \, dw \wedge d\bar w$$

Know $\dfrac{\partial u}{\partial \bar z_1} = \alpha_1$ ✓.

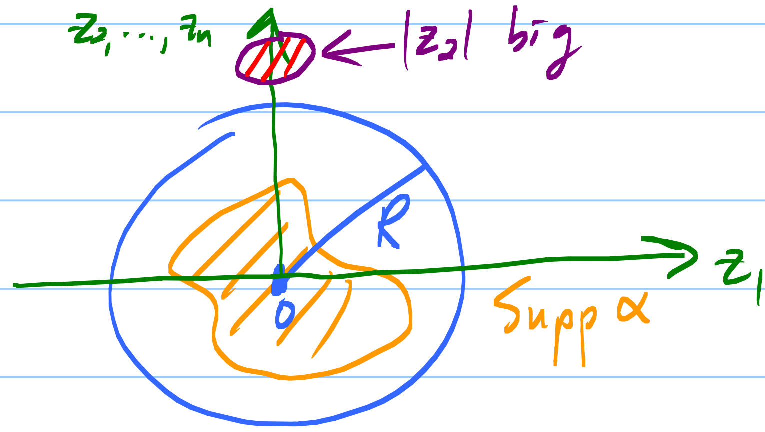
Also
$$\frac{\partial u}{\partial \bar z_i} = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \alpha_1}{\partial \bar z_i}(w, z_2, \dots, z_n)}{w - z_1} \, dw \wedge d\bar w$$

$(i \geq 2)$
$$= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\frac{\partial \alpha_i}{\partial \bar z_1}(w, z_2, \dots, z_n)}{w - z_1} \, dw \wedge d\bar w$$

$$= \alpha_i - \int_{bD_{Big}(0)} \frac{\alpha_i(w)}{w - z_1} \, dw = \alpha_i$$

$\alpha_i(w) = 0$ on big circle

So $\bar\partial u = \alpha$.



$$u(z_1, \dots, z_n) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\alpha_1(w, z_2, \dots, z_n)}{w - z_1} \, dw \wedge d\bar w$$

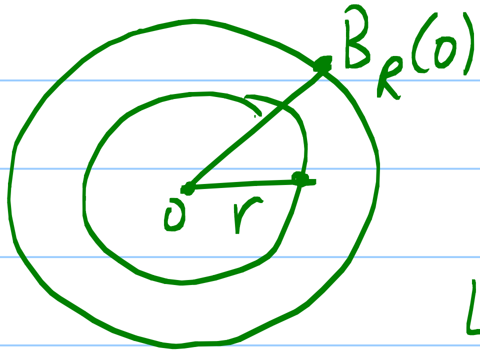
Easy to see $u \equiv 0$ on .

But u analytic outside $D_R(0)$ because $\bar\partial u = 0$ there. Identity Thm says $u \equiv 0$ on $D_R(0)^c$.

<u>Note</u>: In $\mathbb{C}^1$, to solve $\frac{\partial u}{\partial \bar z} = v$, $v \in C_0^\infty(\mathbb{C})$ with $u \in C_0^\infty(\mathbb{C})$, need $\iint_{\mathbb{C}} v\, z^n \, dz \wedge d\bar z = 0$ for all n.

<u>Hartog's Thm 2</u>: Analytic fcns of several complex variables cannot have isolated singularities.

<u>Why</u>:



f analytic on $B_R(0) - B_r(0)$, $0 < r < R$.

Let $\chi \in C^\infty(B_R(0))$ such that $\chi \equiv 0$ on $D_{\rho_1}(0)$ for some ρ_1 with $r < \rho_1 < R$, and $\chi \equiv 1$ on $|x| > \rho_2$ where $r < \rho_1 < \rho_2 < R$. Let
$$\alpha = \begin{cases} \bar\partial(\chi f) & r < |z| < R \\ 0 & |z| \le r. \end{cases}$$

$\alpha_j \in C_0^\infty(D_R(0))$, satisfies (*). $\exists\, u \in C_0^\infty(B_R(0))$ with $\bar\partial u = \alpha$. Now $f - u$ is analytic on $B_R(0)$ and extends f to $B_R(0)$!