Weierstraß Thm: Ω a domain in $\mathbb{C}$ and $\{a_n\}_{n=1}^{\infty}$ discrete points in Ω and $m_n \in \mathbb{N}$. $\exists$ f analytic on Ω with zero of order m_n at a_n for each n and no other zeroes.

Pf in case Ω is simply connected:

Idea: Suppose we have f.

$$f = \exp(\log f) = \exp\left(\int_{\gamma_a^z} \frac{f'}{f}\, dw\right)$$

$$= \exp\left(\int_{\gamma_a^z} \underbrace{\frac{m_n}{w-a_n} + H(w)}_{\text{Hmmm. M-L Idea!}}\, dw\right) \text{ near } a_n.$$

Make it work: M-L Thm yields meromorphic g on Ω with simple poles at each a_n with residue m_n at a_n. Try to "define"

$$f(z) = \exp\left(\int_{\gamma_a^z} g(w)\, dw\right),$$

where a is a fixed point in $\Omega - \{a_n\}_1^{\infty}$ and γ_a^z is a curve in Ω from a to z avoiding $\{a_n\}_1^{\infty}$.

Claim: f is well defined on $\Omega - \{a_n\}_1^{\infty}$.

Suppose γ_a^z and $\tilde{\gamma}_a^z$ are two such curves.

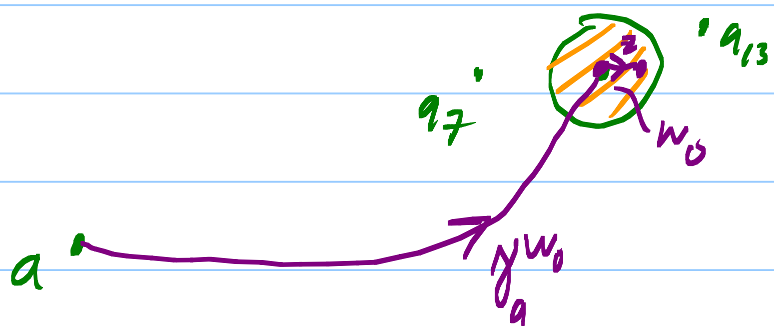
$$\frac{f(z)}{\tilde{f}(z)} = \exp\Big(\underbrace{\int_{\gamma_a^z} g\,dw - \int_{\tilde{\gamma}_a^z} g\,dw} \Big)$$

$$\int_\Gamma g\,dw = 2\pi i \sum \mathrm{Ind}_{a_n}\Gamma \, \underbrace{\mathrm{Res}_{a_n} g}_{m_n}$$

$$= 2\pi i \,(\text{integer})$$

where Γ closed curve. But $\exp(\nearrow) = 1$.

<u>Claim</u>: f is analytic on $\Omega - \{a_n\}_1^\infty$.



$$f = \exp\Big(\underbrace{\int_{\gamma_a^{w_0}} g\,dw}_{\text{const}} + \underbrace{\int_{L_{w_0}^{z}} g\,dw}_{\text{analytic antiderivative for } g \text{ on disc.}} \Big)$$

<u>Finally</u>, f has a removable sing at each a_n.

$$f(z) = \exp\Big(\underbrace{\int_{\gamma_a^{w_0}} g\, dw}_{\text{const}} + \int_{L_{w_0}^{z}} \frac{m_n}{w-a_n} + \underbrace{H_n(w)}_{\text{analytic on disc}}\, dw \Big)$$

$$= \exp\big(C + m_n \log(z-a_n) + h_n(z) \big)$$

$$= (z-a_n)^{m_n} \exp(\text{analytic}). \quad \text{Done!}$$

__(M-L + W):__ Given $\{a_n\}_{n=1}^{\infty}$ in Ω and finitely many Taylor coeff at a_n, then $\exists$ analytic f on Ω having those coeff.

__Cor:__ If f is meromorphic on a domain Ω, then there are analytic functions g and h on Ω such that $f = g/h$.

__Hartog's Thm:__ If f is analytic on $B_R(0) - B_r(0)$, then f extends analytically to $B_R(0)$. ($\mathbb{C}^n$, $n > 1$).

Not possible in $\mathbb{C}$! Example $\Omega = D_1(0)$. Take a seq $\{a_n\}$ like this: N equidistributed points on $C_{1-\frac{1}{N}}$. Every point on $|z| = 1$ is a limit pt of $\{a_n\}$. W-thm gives f with zeroes at a_n that cannot be continued to a bigger open set.

Back to Cauchy Transform for a C^∞ smoothly bounded domain Ω in $\mathbb{C}$.

$$\mathcal{C}u(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z}\,dw$$ u defined on $b\Omega$

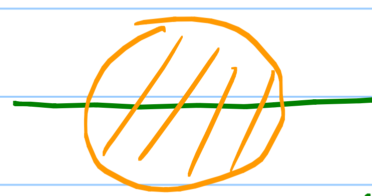
$$U(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{U(w)}{w-z}\,dw + \frac{1}{2\pi i}\iint_{\Omega} \frac{\frac{\partial U}{\partial \bar{w}}}{w-z}\,dw\wedge d\bar{w}$$

$U \in C^\infty(\bar{\Omega})$.

<u>Thm</u>: If $v \in C^\infty(\bar{\Omega})$, then $\frac{1}{2\pi i}\iint_{\Omega} \frac{v(w)}{w-z}\,dw\wedge d\bar{w}$ is a sol^n to $\frac{\partial u}{\partial \bar{z}} = v$ in $C^\infty(\bar{\Omega})$.

<u>Plan</u>: To see $\mathcal{C}: C^\infty(b\Omega) \to C^\infty(b\Omega)$, extend $u \in C^\infty(b\Omega)$ to $U \in C^\infty(\bar{\Omega})$ and apply formulas above. See $\mathcal{C}u = U$ – smooth thing. Finally, restrict $\mathcal{C}u$ to $b\Omega$.

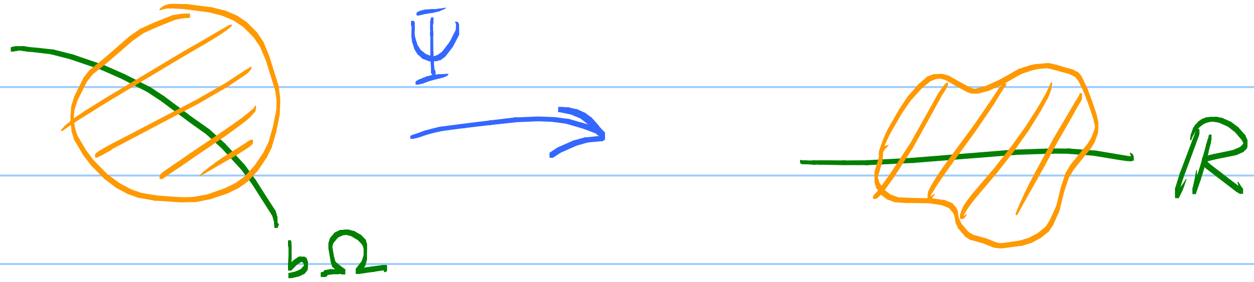
<u>How to cook up U</u>:



$b\Omega = \mathbb{R}$

$u(x)$ in $C^\infty(\mathbb{R})$. Take $U(x,y) = u(x)$

on disc. Easy!



$$\tilde{U} = \left[\text{extend}\left(u\left(\Psi^{-1}\right)\right)\right] \circ \Psi$$ $\leftarrow$ local $\tilde{U}$ near $b\Omega$.

Get global $\tilde{U}$ via a partition of unity.

<u>Thm</u>: $\mathcal{E} : C^\infty(b\Omega) \to C^\infty(\bar{\Omega})$
$\to C^\infty(b\Omega)$.