$$U(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{U(w)}{w-z}\,dw + \frac{1}{2\pi i}\iint_{\Omega} \frac{\frac{\partial U}{\partial \bar w}}{w-z}\, dw\wedge d\bar w$$

Got $\Phi \in C^\infty(\bar\Omega)$ with $\Phi|_{b\Omega} = 0$ and $\frac{\partial U}{\partial \bar w} - \frac{\partial \Phi}{\partial \bar w}$ vanishes to order N on $b\Omega$.

<u>Cauchy Transform:</u> Given $u \in C^\infty(b\Omega)$, extend u to be $U \in C^\infty(\bar\Omega)$.

Ψ

$$U - \Phi = \frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z}\,dw + \frac{1}{2\pi i}\iint_{\Omega} \frac{\frac{\partial U}{\partial \bar w} - \frac{\partial \Phi}{\partial \bar w}}{w-z}\, dw\wedge d\bar w$$

Restrict to $b\Omega$:

$$\boxed{u = \mathcal{C}u + \frac{1}{2\pi i}\iint_{\Omega} \frac{\Psi(w)}{w-z}\, dw\wedge d\bar w}$$

<u>Thm</u>: $\mathcal{C} : C^{s+M}(b\Omega) \longrightarrow C^s(b\Omega)$ $\quad M = M(s)$.

$\mathcal{C} : C^\infty(b\Omega) \longrightarrow C^\infty(b\Omega)$.

Estimate: $\quad \|\mathcal{C}u\|_{C^s(b\Omega)} \le C\|u\|_{C^{s+M}(b\Omega)}$

Words: An operator T is "local" if $u \equiv 0$ on open set $\mathcal{O} \Rightarrow Tu \equiv 0$ on $\mathcal{O}$.

Fact: The Cauchy Transform is "pseudo local." Given open subset $\mathcal{O}$ of $b\Omega$ and $u \in L^2(b\Omega)$. If u is C^∞ on $\tilde{\mathcal{O}}$, then $\mathcal{C}u$ is C^∞ on $\mathcal{O}$, and there is an estimate

$$\|\mathcal{C}u\|_{C^s(\mathcal{O})} \le C\left(\|u\|_{C^{s+m}(\tilde{\mathcal{O}})} + \|u\|_{L^2(b\Omega)} \right)$$

$$(\mathcal{O} \subset\subset \tilde{\mathcal{O}} \subset b\Omega).$$

Why: Cook up $\chi \in C^\infty(b\Omega)$ with $\chi \equiv 1$ on $\tilde{\tilde{\mathcal{O}}}$, $\bar{\mathcal{O}} \subset \tilde{\tilde{\mathcal{O}}}$ and $\chi \in C_0^\infty(\tilde{\mathcal{O}})$.

$$\bar{\mathcal{O}} \subset \tilde{\tilde{\mathcal{O}}} \subset\subset \tilde{\mathcal{O}}$$

$$\mathcal{C}u = \mathcal{C}\Big[\underset{\substack{\uparrow \\ C_0^\infty(\mathcal{O})}}{\chi u} + \underbrace{(1-\chi)u}_{\substack{\text{zero} \\ \text{near } \bar{\mathcal{O}}}}\Big]$$

$$= \underbrace{\mathcal{C}(\chi u)}_{\text{in } C^\infty(b\Omega)} + \underbrace{\frac{1}{2\pi i}\int_{b\Omega - \tilde{\tilde{\mathcal{O}}}} \frac{(1-\chi)u}{w-z}\, dw}_{\substack{\text{can diff under } \int \\ \text{and let } z \to b\Omega \text{ in } \mathcal{O}.}}$$

$$= \left(\underset{\text{term}}{\text{local estimate}}\right) + \left(\underset{\text{term}}{\text{global estimate}}\right)$$

Remark: Given h analytic on Ω and continuous on $\overline{\Omega}$. We have shown that if $h|_{b\Omega}$ is C^∞ on $b\Omega$, then $h \in C^\infty(\overline{\Omega})$.

Same question for harmonic functions is trickier, but true.

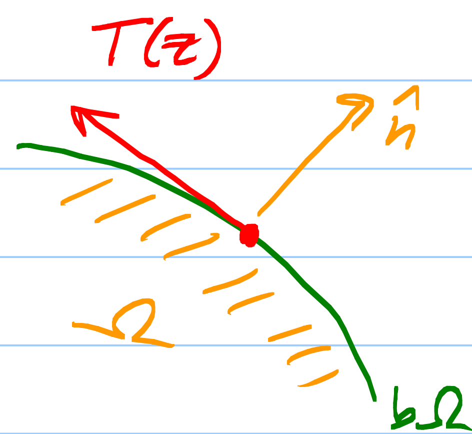
Adjoint of $\mathcal{C}$: $\quad \langle u, v \rangle_{b\Omega} = \int_{b\Omega} u \bar{v} \, ds$

arc length

defined on $L^2(b\Omega)$.

$$\|u\|_{L^2(b\Omega)} = \sqrt{\langle u, u \rangle}.$$

Unit Complex Tangent function T:



$b\Omega$: $\quad z(t)$

$$T(z(t)) = \frac{z'(t)}{|z'(t)|} \in \mathbb{C}$$

Facts: $\quad dz = T\, ds = \dfrac{z'(t)}{|z'(t)|} \sqrt{\dfrac{dx^2}{dt} + \dfrac{dy}{dt^2}}\, dt$ ✓

$$d\bar{z} = \bar{T}\, ds$$

$$ds = T\,d\bar z = \bar T\,dz$$

Theorem: $\mathcal{C}^* v = v - \bar T\,\overline{\mathcal{C}(\bar v \bar T)}$

Pf: u and v in $C^\infty(b\Omega)$.

$$\langle \mathcal{C}u, v\rangle = \left\langle u - \frac{1}{2\pi i}\iint \frac{\Psi(w)}{w-z}\,dw\wedge d\bar w,\ v\right\rangle$$

$$= \langle u, v\rangle - \int_{z\in b\Omega}\left(\frac{1}{2\pi i}\iint_{w\in\Omega}\frac{\Psi(w)}{w-z}\,dw\wedge d\bar w\right)\overline{v(z)}\,ds$$

Big Fact: $\dfrac{\Psi(w)\,v(z)}{w-z}$ is nice on $\Omega\times b\Omega$

Fubini: baby version!

$$= \langle u, v\rangle - \iint_{w\in\Omega}\Psi(w)\underbrace{\left(\frac{1}{2\pi i}\int_{z\in b\Omega}\frac{\overline{v(z)}}{w-z}\underbrace{\overline{T(z)}\,dz}_{ds}\right)}_{-\mathcal{C}(\bar v\bar T)(w)}dw\wedge d\bar w$$

$$= \langle u, v\rangle + \iint_{w\in\Omega}\underbrace{\frac{\partial}{\partial\bar w}(\bar U - \Phi)}_{\Psi}\left[-\mathcal{C}(\bar v\bar T)\right]d\bar w\wedge dw$$

$$= \langle u, v \rangle - \iint_{w \in \Omega} \frac{\partial}{\partial \bar{w}} \left[(\bar{v} - \bar{\Phi}) \mathcal{C}(\bar{v}\bar{T}) \right] d\bar{w} \wedge dw$$

Complex Green's

$$= \langle u, v \rangle - \int_{b\Omega} \underbrace{(\bar{v} - \bar{\Phi})}_{\substack{u - 0 \\ \text{on } b\Omega}} \mathcal{C}(\bar{v}\bar{T}) \underbrace{dw}_{T\,ds}$$

$$= \langle u, v \rangle - \int_{b\Omega} u \, \overline{\mathcal{C}(\bar{v}\bar{T})} \, \bar{\bar{T}} \, ds$$

$$= \langle u, v \rangle - \langle u, \overline{\mathcal{C}(\bar{v}\bar{T})} \, \bar{T} \rangle$$

$$= \langle u, \mathcal{C}^* v \rangle \quad \text{where} \quad \mathcal{C}^* v = v - \bar{T} \, \overline{\mathcal{C}(\bar{v}\bar{T})}.$$

Kerzman-Stein: The Cauchy Transform is "almost" self adjoint!

$\underbrace{\mathcal{C} - \mathcal{C}^*}_{A}$ is a smoothing operator:

$$Au(z) = \int_{w \in b\Omega} A(z, w) \, u(w) \, ds$$

where $A(z,w) \in C^\infty(b\Omega \times b\Omega)$.

A is better than $\mathcal{C}$ or $\mathcal{C}^*$!

$\mathcal{A}$ Kerzman-Stein operator

$A(z,w)$ Kerzman-Stein kernel.