Kerzman-Stein Operator: $\mathcal{A} = \mathcal{C} - \mathcal{C}^*$

where $\mathcal{C}^* v = v - \bar{T}\,\overline{\mathcal{C}(\bar{v}\bar{T})}$

<u>Thm</u>: $(\mathcal{A}u)(z) = \int_{w \in b\Omega} A(z,w)\, u(w)\, ds$

where $A(z,w)$ is in $C^\infty(b\Omega \times b\Omega)$.

So $\mathcal{A}: L^2(b\Omega) \to C^\infty(b\Omega)$, i.e., $\mathcal{A}$ is a <u>smoothing</u> operator.

<u>Hardy space</u>: Let $A^\infty(b\Omega)$ denote the space of fuctions on $b\Omega$ which are the boundary values of analytic functions on Ω in $C^\infty(\bar{\Omega})$. The Hardy Space $H^2(b\Omega)$ is the closure of $A^\infty(b\Omega)$ in $L^2(b\Omega)$.

<u>Def^n</u>: The Szegő projection P is the orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$.

<u>Lemma</u>: If $H \in H^2(b\Omega)$, then $\bar{H}\bar{T} \perp H^2(b\Omega)$.

<u>Pf</u>: $h, H \in H^2(b\Omega)$. Then $\exists\ h_j, H_j$ in $A^\infty(b\Omega)$ with $h_j \xrightarrow[L^2]{} h$ and $H_j \xrightarrow[L^2]{} H$. Now

$$\langle h, \overline{H}\,\overline{T}\rangle = \int_{b\Omega} h\,\overline{\overline{H}\,\overline{T}}\,ds = \int_{b\Omega} h\,H\,\underbrace{T\,ds}_{dz}$$

$$= \lim_{j\to\infty} \underbrace{\int_{b\Omega} h_j H_j\,dz}_{=0 \text{ by Cauchy Thm.}} = 0$$

<u>Remark</u>: Later: $H^2(b\Omega)^\perp = \{\overline{H}\,\overline{T} : H \in H^2(b\Omega)\}$.

<u>Kerzman-Stein Identity</u>: $P(I+A) = \mathcal{C}$

<u>Pf</u>: $(I+A)u = u + Au = u + (\mathcal{C}u - \mathcal{C}^* u)$

$$= u + \mathcal{C}u - \left(u - \overline{T}\,\overline{\mathcal{C}(\bar{u}\,\overline{T})}\right)$$

$$= \mathcal{C}u + \overline{\mathcal{C}(\bar{u}\,\overline{T})}\,\overline{T}$$

$$= \underset{\uparrow \atop P(I+A)u}{h} + \underbrace{\overline{H}\,\overline{T}}_{\perp H^2(b\Omega)}$$

$P(I+A)u$ ✓

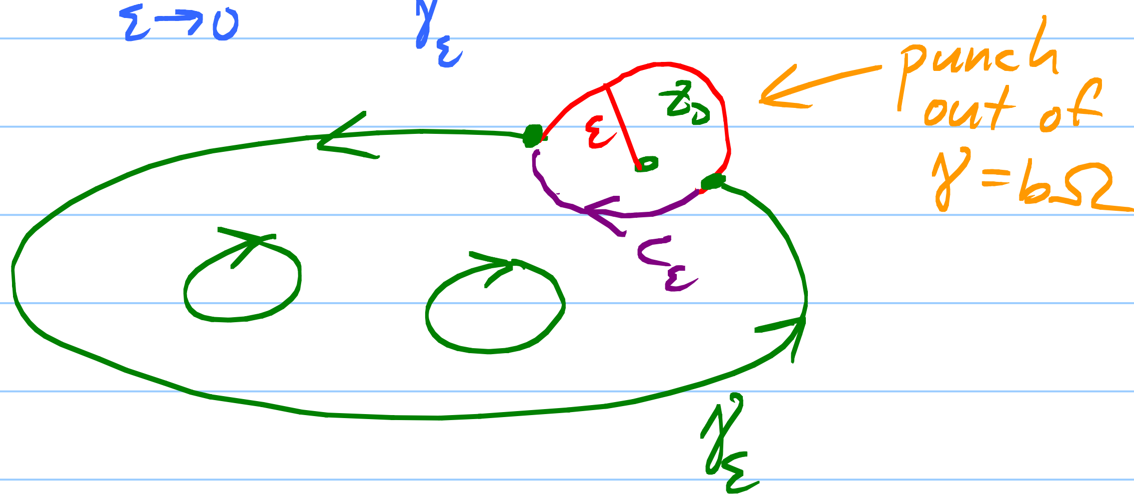
<u>Note</u>: for now, we have only proved K-S for $u \in C^\infty(b\Omega)$.

<u>Plemelj formula</u>: $u \in C^\infty(b\Omega)$, $z_0 \in b\Omega$

$$(\mathcal{C}u)(z_0) = \frac{1}{2}u(z_0) + \underbrace{P.V.\ \frac{1}{2\pi i}\int_{b\Omega} \frac{u(z)}{z-z_0}\,dz}_{(\mathcal{H}u)(z_0)}$$

where

$$(\mathcal{H}u)(z_0) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{u(z)}{z-z_0}\,dz$$



Pf: 1) Plemelj true if $u(z_0)=0$. $PV\int = \int$ ✓

2) true if $u \equiv c$, a const. $\Gamma = \gamma_\varepsilon \cup C_\varepsilon$

$$0 = \frac{1}{2\pi i}\int_\Gamma \frac{c}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{c}{z-z_0}\,dz + \underbrace{\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{c}{z-z_0}\,dz}_{-\frac{1}{2}c}$$

Cauchy Thm

$z(t) = z_0 + \varepsilon e^{it}$

Finally, $\mathcal{C}c \equiv c$. Done!

3) General case. Replace u by $u(z)-u(z_0)$ and apply cases 1 & 2. ✓

<u>Lemma</u>: $\mathcal{A}u = \mathcal{H}u + \overline{T}\,\overline{\mathcal{H}(\bar{u}\overline{T})}$

<u>Pf</u>: $\mathcal{A}u = \mathcal{C}u - \mathcal{C}^{*}u = \mathcal{C}u - \left(u - \overline{T}\,\overline{\mathcal{C}(\bar{u}\overline{T})}\right)$

$$= \left[\tfrac{1}{2}u + \mathcal{H}u\right] - \left(u - \overline{T}\,\overline{\left(\tfrac{1}{2}\bar{u}\overline{T} + \mathcal{H}(\bar{u}\overline{T})\right)}\right)$$

$$= \mathcal{H}u + \overline{T}\,\overline{\mathcal{H}(\bar{u}\overline{T})} \ \checkmark$$

<u>Cor</u>:
$$\mathcal{A}u = \text{P.V.}\int_{w\in b\Omega} A(z,w)\, u(w)\, ds$$

where
$$A(z,w) = \frac{1}{2\pi i}\left(\frac{T(z)}{w-z} - \frac{\overline{T(w)}}{\bar{w}-\bar{z}}\right)$$

<u>Pf</u>: Just write it out! $\left(\overline{T}\,dz = ds\right)$

<u>Big Fact</u>: $A(z,w)$ does not blow up at $z=w$!

Calculus lemma: If $\mathbb{X}(s,t)$ is C^∞ on $[a,b]\times[c,d]$ and $\mathbb{X}(s,s)=0$, then $\mathbb{X}(s,t)=(s-t)\mathbb{Y}(s,t)$ where $\mathbb{Y}$ is C^∞ on $[a,b]\times[c,d]$.

Pf: $\mathbb{X}(s,t) = \mathbb{X}(s,t) - \underbrace{\mathbb{X}(s,s)}_{\overset{\shortparallel}{0}}$

$$= \int_s^t \frac{\partial \mathbb{X}}{\partial u}(s,u)\,du \qquad u = s+v(t-s)$$

$$= \int_0^1 \frac{\partial \mathbb{X}}{\partial u}(s, s+v(t-s))\,(t-s)\,dv$$

$$= (t-s)\underbrace{\int_0^1 \frac{\partial \mathbb{X}}{\partial u}(s, s+v(t-s))\,dv}_{\text{smooth!}}$$

Assume $A(z,w)$ is in C^∞ of $b\Omega\times b\Omega$ for now.

Thm: $\mathcal{C}: L^2(b\Omega)\circlearrowleft$ bounded!

Pf: $\mathcal{C} = P(I+A)$

Thm: $\mathcal{C}^* u = u - \bar{T}\,\overline{\mathcal{C}(\bar{u}\bar{T})}$ is the L^2 adjoint.

Pf: $\langle \mathcal{C}u, v\rangle = \langle u, \mathcal{C}^* v\rangle$ when $u, v \in C^\infty(b\Omega)$.

If $u, v \in L^2(b\Omega)$, approx by C^∞ fcns and take limits.