$$A(z,w) = \frac{1}{2\pi i}\left(\frac{T(w)}{w-z} - \frac{\overline{T(z)}}{\bar{w}-\bar{z}}\right)$$

$b\Omega$ parametrized by $z(t)$ where t is arc length.

Then $T(z(t)) = \dfrac{z'(t)}{\underbrace{|z'(t)|}_{=1}} = z'(t)$.

$$A(\underbrace{z(t)}_{z}, \underbrace{z(s)}_{w}) = \frac{1}{2\pi i}\left(\frac{z'(s)}{z(s)-z(t)} - \frac{\overline{z'(t)}}{\overline{z(s)}-\overline{z(t)}}\right) = (*)$$

Write $Q(s,t) = z(s) - z(t) = (s-t)\,\mathbb{X}(s,t)$

because Q is C^∞ and $Q(s,s) \equiv 0$.

$$\frac{z(s)-z(t)}{s-t} \to z'(t) \quad \text{as} \quad s \to t.$$

So $\mathbb{X}(t,t) = z'(t)$.

$$\text{Now} \quad (*) = \frac{1}{2\pi i}\,\frac{1}{s-t}\underbrace{\left[\frac{z'(s)}{\mathbb{X}(s,t)} - \frac{\overline{z'(t)}}{\overline{\mathbb{X}(s,t)}}\right]}_{\mathbb{Y}(s,t)}$$

$\mathbb{Y}(s,t)$ is C^∞ for s near t because $\mathbb{X}(s,t) \neq 0$ there. Aha! $\mathbb{Y}(t,t) = \dfrac{1}{2\pi i}\left(\dfrac{z'(t)}{z'(t)} - \dfrac{\overline{z'(t)}}{\overline{z'(t)}}\right) = 0$.

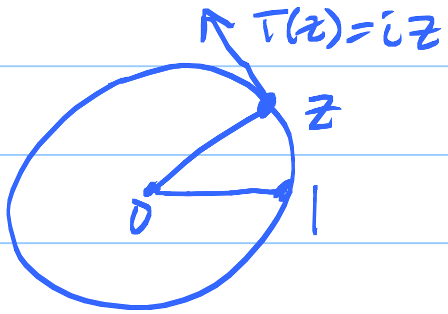
So $\mathbb{Y}(s,t) = (s-t)\,\mathbb{W}(s,t)$ where $\mathbb{W}$ is C^∞.

Now $(*) = \frac{1}{2\pi i}\,\mathbb{W}(s,t)$ is seen to be smooth.

Conclude $A(z,w)$ is in $C^\infty(b\Omega \times b\Omega)$.

Remark: Since $A(z,w) = -\overline{A(w,z)}$ off the diagonal, same holds on diag by continuity. So $A(z,z) \equiv 0$.

Fact: $A(z,w) \equiv 0$ if Ω is a disc.



Write it down.

Remark: Can drop P.V. from formula

$$Au = \int_{w \in b\Omega} A(z,w)\, u(w)\, ds.$$

Theorem: $P: C^\infty(b\Omega) \circlearrowleft$

Proof: K-S Formula $\quad P(I + A) = \mathcal{C}$

Take L^2 adjoint. Note: $A^* = (\mathcal{C} - \mathcal{C}^*)^* = \mathcal{C}^* - \mathcal{C}$
$= -A$

$P^* = P$ since P projection.

$$\begin{array}{ll} \text{K-S} & P + PA = \mathcal{C} \\ * & P + A^*P = \mathcal{C}^* \\ & P - AP = \mathcal{C}^* \end{array}$$

$-$ $\quad PA + AP = \underbrace{\mathcal{C} - \mathcal{C}^*}_{A}$

$$\boxed{PA + AP = A}$$

So $\underline{PA} = -AP + A = A(I - P)$, a smoothing oper!

K-S: $P(I + A) = C$

$P + PA = C$

$P = C - \underline{PA}$ takes $C^\infty(b\Omega) \circlearrowleft$

<u>Theorem</u>: A function $u \in L^2(b\Omega)$ has an orthogonal decomposition of the form $u = h + \overline{HT}$ where $h, H \in H^2(b\Omega)$ are uniquely determined:

$$h = Pu, \quad \overline{H} = \overline{P(\overline{u}\,\overline{T})}$$

<u>Pf</u>: Let $h = Pu$. Then $v = u - h \perp H^2(b\Omega)$.

<u>Lemma</u>: If $v \perp H^2(b\Omega)$, then $C^* v = 0$.

So $v = \overline{T}\,\overline{C(\overline{v}\,\overline{T})} = \overline{HT}$.

<u>Pf of Lemma</u>: Suppose $v \perp H^2(b\Omega)$ and $\varphi \in C^\infty(b\Omega)$.

$0 = \langle C\varphi, v \rangle = \langle \varphi, C^* v \rangle$ for any such φ.

So $C^* v = v - \overline{T}\,\overline{C(\overline{v}\,\overline{T})} = 0$, and $v = \overline{HT}$

where $H = C(\overline{v}\,\overline{T})$.

Pf of Thm: $u-h \perp H^2$. So $u-h=\overline{H\,T}$.

Now $u = h + \overline{H\,T}$.

Hit with P: $Pu = \underbrace{Ph}_{h} + \underbrace{P(\overline{H\,T})}_{0} = h$ ✓

Next $u = h + \overline{H}\,\overline{T}$ ← mult. by T. Use $\overline{T}\cdot T \equiv 1$.

$uT = hT + \overline{H}$

$\overline{u}\,\overline{T} = \overline{h}\,\overline{T} + H$ ← Take conjugate.

Finally $P(\overline{u}\,\overline{T}) = H$. ✓

On $D_1(0)$: $u = h + \overline{H}\,\overline{T} = \underbrace{h + \overline{H\,(iz)}}$

Harmonic extension of u to $D_1(0)$.

Aha! Sol^n is in $C^\infty(\overline{D_1(0)})$.

The Szegö kernel: Take $h \in A^\infty(b\Omega)$.

$$h(a) = \frac{1}{2\pi i}\int_{z\in b\Omega} \frac{h(z)}{z-a}\,dz = \int_{z\in b\Omega} h(z)\,\frac{1}{2\pi i}\frac{T(z)}{z-a}\,ds$$

$= \langle h, C_a\rangle$ where $C_a(z) = \frac{1}{2\pi i}\frac{\overline{T(z)}}{(\bar a-\bar z)}$

is the "Cauchy kernel".

$$h(a) = \langle h, C_a \rangle = \langle h, \underbrace{PC_a}_{\substack{\in H^2(b\Omega), \\ \in A^\infty(b\Omega).}} \rangle$$

Szegö kernel: $S_a = PC_a$

Alternative way: $h(a) = \langle h, C_a \rangle$.

$$|h(a)| \le \|h\|_{L^2} \, \|C_a\|_{L^2} \le c\|h\|$$

So point evaluation at a is a continuous linear functional on $H^2(b\Omega)$. Riesz Rep Thm gives $\exists\, S_a$.

Next: $\underbrace{C_a}_{\frac{1}{2\pi i}\frac{\bar{T}}{\bar{a}-\bar{z}}} = \underbrace{PC_a}_{S_a} + \overline{H_a}\,\bar{T}$

$$S_a(z) = \frac{1}{2\pi i}\frac{\overline{T(z)}}{\bar{z}-\bar{a}} - \overline{H_a}\,\bar{T}$$

$$= \overline{\frac{1}{2\pi i}\left(\frac{1}{z-a} - H_a\right)}\;\bar{T}$$

$$= \overline{G_a}\;\bar{T}$$

very interesting analytic function with a simple pole at a