$$P = \mathcal{C} - \underbrace{A(I-P)}_{\text{smoothing operator}}$$

P and $\mathcal{C}$ differ by a smoothing operator

Szegő kernel: $\quad S_a = P\mathcal{C}_a \qquad \underbrace{\mathcal{C}_a(z) = \frac{1}{2\pi i}\,\frac{\overline{T(z)}}{\bar a - \bar z}}_{\text{Cauchy kernel}}$

$$h(a) = \underbrace{\langle h, \mathcal{C}_a\rangle}_{\text{Cauchy Integral formula}} = \langle h, \underbrace{P\mathcal{C}_a}_{S_a}\rangle$$

Fact: $\quad \langle u, S_a\rangle = \langle Pu, S_a\rangle = (Pu)(a)$

$$(Pu)(a) = \int_{z\in b\Omega} u(z)\,\underbrace{\overline{S_a(z)}}_{S(a,z)}\,ds$$

The Szegő kernel is the kernel for P.

Garabedian kernel: $\quad \mathcal{C}_a = \underset{\underset{P\mathcal{C}_a}{\uparrow}}{S_a} + \underset{\underset{P^{\perp}\mathcal{C}_a}{\uparrow}}{\overline{H_a T}}$

$$\frac{1}{2\pi i}\,\frac{\overline{T}}{\bar a - \bar z} = S_a + \overline{H_a T} \quad \leftarrow \text{Take conjugate}$$

$$\overline{S_a} = \frac{1}{i}\underbrace{\left(\frac{1}{2\pi(z-a)} - iH_a\right)}_{L_a}T \qquad \boxed{\overline{S_a} = \frac{1}{i}L_a T}$$

L_a Garabedian kernel.

Notation: $S_a(z) = S(z,a)$

$L_a(z) = L(z,a)$

Facts: 1) S_a and L_a extend C^∞ smoothly to $b\Omega$.

2) $L_a(z)$ has a simple pole, residue $\frac{1}{2\pi}$ at a.
$L_a \in C^\infty(\bar{\Omega} - \{a\})$.

3) $S(z,a)$ is analytic in z and anti-analytic in a on $\Omega \times \Omega$.

4) $S(z,a) = \overline{S(a,z)}$

5) $L(z,a) = -L(a,z)$
$L(z,a)$ is analytic in z and a on $\Omega \times \Omega$.

6) $L(z,a) \neq 0$ if $z \neq a$, $z,a \in \bar{\Omega}$.

7) If Ω is simply connected, then $S(z,a) \neq 0$ for all $(z,a) \in \bar{\Omega} \times \Omega$. The Riemann mapping function f_a is given by $\frac{S(z,a)}{L(z,a)}$.

Book www.math.purdue.edu/~bell/MA693/ 1-20.pdf
21-40.pdf

8) $S(a,a) \in \mathbb{R}$ and > 0.

<u>Pfs:</u> 3) $\quad S_a(a) = \langle S_a, S_a \rangle = \|S_a\|_{L^2}^2 > 0$

(Note $S_a \not\equiv 0$ because $\langle 1, S_a \rangle = 1$.)

4) $\quad S_a(b) = \langle S_a, S_b \rangle = \overline{\langle S_b, S_a \rangle} = \overline{S_b(a)}$.

So $S(z,a)$ is anti-analytic in a.

Alternatively:
$$\frac{S_a(z) - S_{a_0}(z)}{\bar{a} - \bar{a}_0} = P\left(\frac{e_a - e_{a_0}}{\bar{a} - \bar{a}_0}\right)$$
$$\longrightarrow P\left(e'_{a_0}\right)$$

<u>Carathèodory's Thm:</u> Ω is a C^∞ smoothly bounded bounded domain $\neq \mathbb{C}$. Given $a \in \Omega$, the Riemann map $f_a : \Omega \to D_1(0)$ with $f_a(a) = 0$ and $f_a'(a) > 0$ is <u>continuous</u> on $\overline{\Omega}$, and $f_a : b\Omega \to \{|z| = 1\}$ 1-1 onto.

<u>Trick:</u> $h \in A^\infty(b\Omega)$.

$$\langle f_a L_a, h \rangle = \int_{b\Omega} f_a L_a \overline{h}\, ds$$

$$= \int_{b\Omega} \frac{1}{f_a}\left(i\,\overline{S_a}\,\overline{T}\right)\overline{h}\, ds$$

$b\Omega$:

$\overline{S_a} = \frac{1}{i} L_a T$

$L_a = i\,\overline{S_a}\,\frac{1}{T}$

$T \cdot \overline{T} = 1$

$f_a \cdot \overline{f_a} = 1$

4

$$= i \int_{b\Omega} \frac{1}{\bar{f}_a} \bar{S}_a \bar{h}\, d\bar{z} = \overline{2\pi i \operatorname{Res}_a\left(\frac{-i S_a h}{f_a}\right)}$$

$$= 2\pi \frac{\overline{S_a(a) h(a)}}{\overline{f'(a)}} = \underbrace{2\pi \frac{S_a(a)}{f'(a)}}_{c} \overline{h(a)}$$

Hmmm.

$$\langle c S_a, h \rangle = \text{same thing!}$$

$$\langle \underbrace{c S_a - f_a L_a}_{\text{must be zero!}}, h \rangle = 0 \quad \forall h \in A^\infty(b\Omega),$$

$$f_a = c \frac{S_a}{L_a}$$

Finally, write out $f_a'(a)$ to see $c = 1$.

<u>Cor</u>: $f_a \in C^\infty(\bar{\Omega})$.

<u>Remark</u>: $\bar{S}_a = \frac{1}{i} L_a T$ Add up arguments as go around $b\Omega$ G.

$$(-\#\text{ zeroes } S_a) = \left[(\#\text{ zeroes } L_a) - (\underset{\substack{\uparrow \\ \text{simple pole at } a}}{1})\right] + \frac{2\pi}{2\pi}$$

by Arg. Princ. $-S\# = +L\#$. Both must $= 0$.

Fact: $f_a'(a) = 2\pi S(a,a)$ because

$$f_a = \frac{S(z,a)}{L(z,a)} \; ; \qquad L_a f_a = S_a$$

$$\underbrace{\mathrm{Res}_a L_a}_{1/2\pi} \; f_a'(a) = S(a,a)$$

Bergman kernel:

$$f_a'(z) = cK(z,a)$$