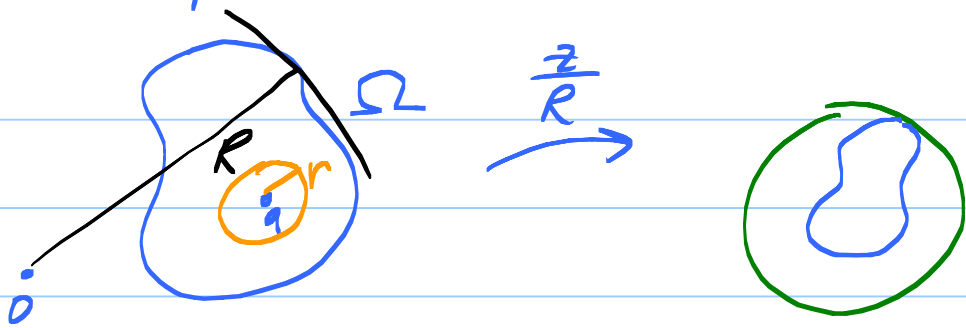
Riemann map: $f_a'(a) = \mathrm{Max}\{|h'(a)| : h : \Omega \to D_1(0)\}$.



$$\frac{1}{r} > f_a'(a) > \frac{1}{R}$$

Cauchy Estimate

<u>Remark</u>: For bounded domains, we can prove R. Map. Thm. by exhausting by C^∞ domains and using Hurwicz, Normal Families, Arg. Princ.

<u>Gen. Case</u>: Step 1: Map first to a bounded region, and then use the above.

$$L(z,a) = \frac{1}{2\pi i(z-a)} + \ell(z,a)$$

$\in C^\infty(\bar\Omega \times \bar\Omega)$

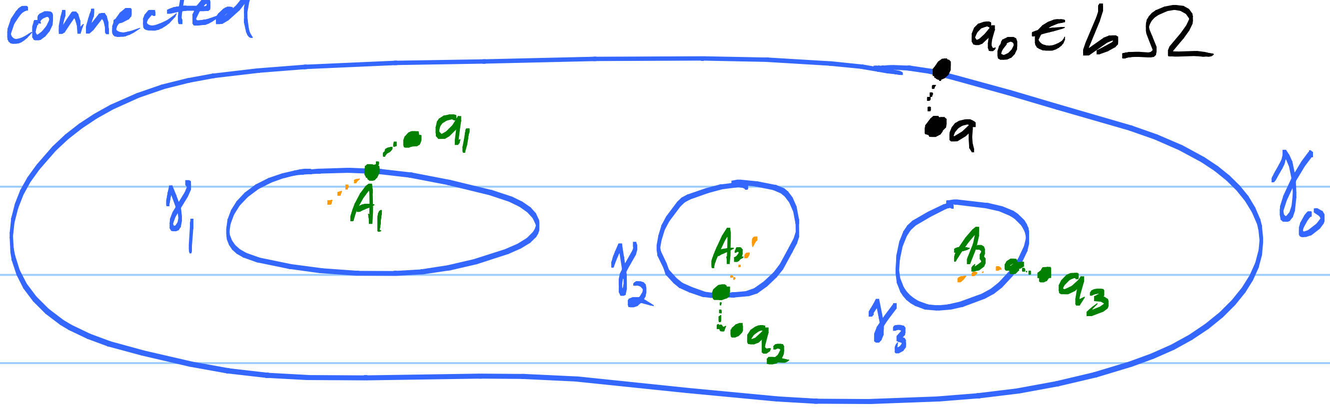
<u>Cor</u>: $L \in C^\infty((\bar\Omega \times \bar\Omega) - \Delta)$ where $\Delta = \{(z,z) : z \in \bar\Omega\}$

$S \in C^\infty((\bar\Omega \times \bar\Omega) - b\Delta)$ where $b\Delta = \{(z,z) : z \in b\Omega\}$

$L(z,a) \neq 0 \quad (z,a) \in \bar\Omega \times \Omega,\ z \neq a.$ ← always true. Proved in s.c.

$S(z,a) \neq 0 \quad (z,a) \in \bar\Omega \times \Omega$ ← only true for s.c.

n-connected



$S(a_j, a) = 0$, a_j zeroes of $S(z,a)$.

$S(z,a)$ has $n-1$ zeroes. As $a \to a_0 \in b\Omega$,

a_j separate and head toward distinct A_j,

one per $\gamma_j \neq \gamma_0$.

$S(A_j, a_0) = 0$. $\quad L(A_j, a_0) = 0$ too!

Solution of the Dirichlet Problem on a bounded C^∞ smooth simply connected domain.

$$S_a\varphi = h + \overline{H}\,\overline{T} \qquad \text{where} \quad h = P(S_a\varphi), \quad H = P(\overline{S_a\varphi\,\overline{T}})$$

$$\varphi = \frac{h}{S_a} + \frac{\overline{H}\,\overline{T}}{S_a} \qquad \overline{S_a} = \frac{1}{i} L_a T$$

$$S_a = -\frac{1}{i}\overline{L_a}\,\overline{T}$$

$$= \frac{h}{S_a} + \frac{\overline{H}\,(-i)}{\overline{L_a}} \qquad \overline{T} = \frac{-i\,S_a}{\overline{L_a}}$$

Aha! $\quad \varphi = g + \overline{G}$ where $g, G \in A^\infty(\Omega)$.

This gives the harmonic extension of φ to Ω.

Note: $H = P(\overline{S_a \varphi \, T}) = P(\frac{1}{i} L_a \overline{\varphi})$.

Thm: Given $\varphi \in C^\infty(b\Omega)$, the harmonic extension of φ is given by $g + \overline{G}$ where

$$g = \frac{P(S_a \varphi)}{S_a} \quad \text{and} \quad G = \frac{P(L_a \overline{\varphi})}{L_a} \quad \text{and}$$

$g, G \in A^\infty(\Omega)$.

Cor: Approximate φ cont on $b\Omega$ by $\varphi_j \in C^\infty(b\Omega)$ and use max princ to get harmonic fcn continuous on $\overline{\Omega}$, harmonic in Ω, and $= \varphi$ on $b\Omega$.

Remark: Sol^n to Dirichlet Prob is

$$u(z) = \int_{w \in b\Omega} p(z,w)\, \varphi(w)\, ds \quad \text{where}$$

$$p(z,w) = \frac{S(z,w)\, S(w,a)}{S(z,a)} + \overline{\frac{S(z,w)\, L(w,a)}{L(z,a)}}$$

is the Poisson Kernel.

Note: $p(z,w)$ does not depend on a. Uniqueness of $p(z,w)$ is easy. Hmmm. Let $a \to z$

above. Pole of $L(z,a)$ at $z=a$ kills second term. See

$$p(z,w) = \frac{S(z,w)S(w,z)}{S(z,z)} = \frac{|S(z,w)|^2}{S(z,z)}$$

Poisson kernel ← Thm. Same in s.c. domains. → Poisson-Szegő kernel

Dirichlet Problem #2: Given $v \in C^\infty(\bar{\Omega})$, find $u \in C^\infty(\bar{\Omega})$ with $\begin{cases} \Delta u = v \\ u|_{b\Omega} = 0. \end{cases}$

Green's Operator: $u = Gv$.

Fact: $\frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} = \frac{1}{4}\Delta$

Know $$u(z) = \frac{1}{2\pi i} \iint_\Omega \frac{v(w)}{w-z}\, dw \wedge d\bar{w}$$

solves $\frac{\partial u}{\partial \bar{z}} = v$ with $u \in C^\infty(\bar{\Omega})$.

Take conjugate and use $\overline{\frac{\partial u}{\partial z}} = \frac{\partial \bar{u}}{\partial \bar{z}}$ to see we can also solve $\frac{\partial u}{\partial z} = v$ with $u \in C^\infty(\bar{\Omega})$.

Step 1: Get u_1 with $\frac{\partial u_1}{\partial \bar{z}} = \frac{1}{4}v$.

Get u_2 with $\frac{\partial u_2}{\partial z} = u_1$. Then

$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} u_2 = \frac{1}{4}\Delta u_2 = \frac{1}{4}V$. So $\Delta u_2 = V$

and $u_2 \in C^\infty(\bar\Omega)$. Finally, get harmonic

U on Ω in $C^\infty(\bar\Omega)$ with boundary values the same as u_2.

$u_2 - U$ solves D. Prob. #2 for v.

Thm: Solution to D. Prob. #2 for $v \in C^\infty(\bar\Omega)$ is in $C^\infty(\bar\Omega)$.

Remark: Solution is unique: $\begin{cases} \Delta u_j = v & , j=1,2 \\ u_j|_{b\Omega} = 0. \end{cases}$

$\Delta(\underbrace{u_1 - u_2}) = 0$, (harmonic).

zero on $b\Omega$. Max Princ $\Rightarrow u_1 - u_2 \equiv 0$.

Def^n: Suppose Ω is a bounded domain in $\mathbb{C}$.

$$L^2(\Omega) = \{ u : \iint_\Omega |u|^2\, dA < \infty \}.$$

$$\langle u, v\rangle = \iint_\Omega u\bar v\, dA.$$

$H^2(\Omega)$ is the closure in $L^2(\Omega)$ of the subspace of analytic functions.

Bergman Proj. $B : L^2(\Omega) \to H^2(\Omega)$ orthog. proj.