www.math.purdue.edu/~bell/MA693/ 1-20.pdf

21-40.pdf

41-80.pdf

book.pdf

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$$B = I - 4\frac{\partial}{\partial z} G \frac{\partial}{\partial \bar z}$$ ← Spencer's formula

in $\mathbb{C}^n$: $B = I - \bar\partial^* N \bar\partial$

Thm: $B : C^\infty(\bar\Omega) \to C^\infty(\bar\Omega)$ if Ω is a bounded C^∞ smooth domain.

Cor: $A^\infty(\Omega)$ is dense in $H^2(\Omega)$.

Pf: Given $h \in H^2(\Omega)$, take $u_j \in C^\infty(\bar\Omega)$ with $u_j \to h$ in $L^2(\Omega)$. Then

$$\underbrace{Bu_j}_{= h_j \in A^\infty(\Omega).} \to Bh = h \text{ in } L^2(\Omega)$$

Cor: The Bergman span, $\mathcal{K}$ = complex linear span of K_a as a ranges over Ω, is dense in $H^2(\Omega)$.

Pf: If not, $\exists\ g \not\equiv 0$ in $H^2(\Omega)$ with $g \perp \mathcal{K}$. [$H^2 = \bar{\mathcal{K}} \oplus \bar{\mathcal{K}}^\perp$ if $\bar{\mathcal{K}} \neq H^2$.]

Hmmm. $g(a) = \langle g, K_a \rangle = 0$ for all $a \in \Omega$.

Cor: Complex linear span of K_a as a ranges over any set $S' \subset \Omega$ with a limit pt in Ω is dense in H^2.

Thm: $\mathcal{K}$ is dense in $A^\infty(\Omega)$.

Pf: $\theta \in C_0^\infty(D_1(0))$, radially symmetric, $\theta \geq 0$,

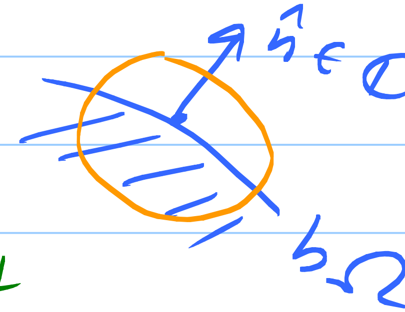
$$\iint \theta \, dA = 1.$$

$$\theta_\varepsilon(z) = \frac{1}{\varepsilon^2}\theta(z/\varepsilon). \qquad \theta_\varepsilon \in C_0^\infty(D_\varepsilon(0)),$$

$$\iint \theta_\varepsilon \, dA = 1.$$

Given $h \in A^\infty(\Omega)$. Know $\exists\, \varphi \in C^\infty(\bar\Omega)$ such that $\varphi|_{b\Omega} = 0$ and $\frac{\partial\varphi}{\partial z}$ agrees with h to arbitrarily high order on $b\Omega$.

$$h = Bh = B\Big(\underbrace{h - \frac{\partial\varphi}{\partial z}}_{\text{vanishes to high order on } b\Omega}\Big) \qquad \frac{\partial\varphi}{\partial z} \perp H^2, \text{ so } B\frac{\partial\varphi}{\partial z} = 0$$



Use partition of unity argument to cook up $\psi_\varepsilon \in C_0^\infty(\Omega)$ close to $\psi = h - \frac{\partial\varphi}{\partial z}$ in $C^\infty(\bar\Omega)$. Say $\operatorname{supp}\psi_\varepsilon \subset \{z : \operatorname{dist}(z, b\Omega) > \varepsilon\}$.

Now $\theta_e * \Psi_\varepsilon$ is in $C_0^\infty(\Omega)$ if $e < \varepsilon$ and can be made as close to Ψ_ε as desired.

$$h = B(\Psi) \approx B(\Psi_\varepsilon) \cong B(\theta_e * \Psi_\varepsilon).$$

But $$\theta_e * \Psi_\varepsilon = \iint_{w \in \mathbb{C}} \theta_e(z-w)\,\Psi_\varepsilon(w)\,dA$$

$$\cong \underbrace{\sum}_{\text{Riemann Sum}} c_j\,\theta_\varepsilon(z-w_j)$$

Now take B of last approximation:

Use $B(\theta_\varepsilon(\cdot - w_j)) = K_{w_j}$. ✓

<u>History</u>: Open question in 1976 in $\mathbb{C}^n$.

$K_a(z) = K(z,a)$. Given $z_0 \in b\Omega$, could $K(z_0, a) \equiv 0$ for $a \in \Omega$? Not if $\mathcal{K}$ is dense in $A^\infty(\Omega)$. $1 \approx \sum c_j K_{a_j}$

<u>Weyl's Lemma</u> (Havin's Lemma):

<u>Def^n</u>: Given $u, v \in L^2(\Omega)$, then u is

a weak solution to $\frac{\partial u}{\partial \bar{z}} = v$ if

$$\langle v, \varphi \rangle = \langle \frac{\partial u}{\partial \bar{z}}, \varphi \rangle = \iint_{\Omega} \frac{\partial u}{\partial \bar{z}} \bar{\varphi} \, dA$$

$$= -\iint u \frac{\partial \bar{\varphi}}{\partial \bar{z}} \, dA = -\langle u, \frac{\partial \varphi}{\partial z} \rangle$$

$$\uparrow \overline{\left(\frac{\partial \varphi}{\partial z}\right)}$$

for all $\varphi \in C_0^\infty(\Omega)$. [Erase grey stuff.]

<u>Weyl's Lemma:</u> If $h \in L^2(\Omega)$ and $\frac{\partial h}{\partial \bar{z}} = 0$ in the weak sense on Ω, then $h \in H^2$, [i.e., $\exists$ H analytic on Ω, $H \in L^2(\Omega)$, and $h = H$ a.e.]

<u>Idea</u>: $h_\varepsilon = \theta_\varepsilon * h$

$$h_\varepsilon(z) = \iint_{w \in \Omega} \theta_\varepsilon(z-w) \, h(w) \, dA$$

$$\frac{\partial h_\varepsilon}{\partial \bar{z}} = \iint_{w \in \Omega} \frac{\partial \theta_\varepsilon}{\partial \bar{z}}(z-w) \, h(w) \, dA$$

$\theta_\varepsilon(\xi)$ $\quad \xi = z - w$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial \theta_\varepsilon}{\partial \bar{\xi}} \overset{1}{\frac{\partial \bar{\xi}}{\partial \bar{z}}} + \frac{\partial \theta_\varepsilon}{\partial \xi} \overset{0}{\frac{\partial \xi}{\partial \bar{z}}}$$

$$= \iint_{w\in\Omega} -\frac{\partial\theta_\varepsilon}{\partial\bar{w}}(z-w)\, h(w)\, dA$$

$$= \left\langle h, -\overline{\frac{\partial\theta_\varepsilon}{\partial\bar{w}}(z-\cdot)} \right\rangle \qquad \overline{\left(\frac{\partial\theta_\varepsilon}{\partial\bar{w}}\right)} = \frac{\partial\bar{\theta}_\varepsilon}{\partial w}$$

$= 0$ because h is a weak sol^n to CR eqns.

h_ε is analytic on $\{z\in\Omega : \text{dist}(z, b\Omega) > \varepsilon\}$.

But $h_\varepsilon \to h$ in L^2 as $\varepsilon \to 0$.

So $h =$ analytic fcn a.e.

Cor: $\left\{ \frac{\partial\varphi}{\partial z} : \varphi\in C_0^\infty(\Omega) \right\}$ is dense in $H^2(\Omega)^\perp$.