Cor: $\left\{ \frac{\partial \varphi}{\partial z} : \varphi \in C_0^\infty(\Omega) \right\}$ is dense in $H^2(\Omega)^\perp$.

Pf: If not dense, $H^{2\perp} = \bar{S} \oplus \bar{S}^\perp$.

If $v \not\equiv 0 \in \bar{S}^\perp$, then $\langle v, \frac{\partial \varphi}{\partial z} \rangle = 0$

for all $\varphi \in C_0^\infty(\Omega)$. Oops! So $v \in H^2$.

But $v \in H^{2\perp}$. $\lightning$.

Transformation of Bergman projection and kernel under conformal maps.

$f : \Omega_1 \to \Omega_2$ biholomorphic map between bounded domains.

Write $f(x+iy) = u(x,y) + i v(x,y)$

$$\det \begin{bmatrix} u_x & \underset{}{\overset{= -v_x}{u_y}} \\ v_x & \underset{= u_x}{v_y} \end{bmatrix} = u_x^2 + v_x^2 = |f'|^2$$

$$f' = u_x + i v_x$$
$$= v_y - i u_y$$

Consequence: Operator $\Lambda : u \longmapsto f' \cdot (u \circ f)$

is an isometry! $\Lambda : L^2(\Omega_2) \to L^2(\Omega_1)$

$\Lambda : H^2(\Omega_2) \to H^2(\Omega_2)$.

Pf: $\|\Lambda u\|^2 = \iint_{\Omega_1} |f'(u\circ f)|^2 \, dA$

$= \iint_{\Omega_1} |\det J_f| \, |(u\circ f)|^2 \, dA \underset{\substack{\uparrow \\ \text{change} \\ \text{of vars} \\ \text{formula}}}{=} \iint_{\Omega_2} |u|^2 \, dA$

$= \|u\|^2$.

Note: Λ preserves analytic fcns. So it is an isometry on Bergman spaces too.

Consequence: $\langle \Lambda u, \Lambda v\rangle = \langle u, v\rangle$ too.

Note: $\Lambda \frac{\partial \varphi}{\partial z} = f'\left(\frac{\partial \varphi}{\partial z}\circ f\right) = \frac{\partial}{\partial z}(\varphi \circ f)$.

Theorem: $B_1 \Lambda = \Lambda B_2$

$B_1(f'(u\circ f)) = f'\left((B_2 u)\circ f\right)$

Pf: Follows from isometry, $H^2 \to H^2$.

or think like this:

$$u = h + v \cong h + \frac{\partial \varphi}{\partial z}$$

$$f'(u \circ f) \cong f'(h \circ f) + \frac{\partial}{\partial z}(\varphi \circ f)$$

$$B_1\left(f'(u \circ f)\right) \cong f'(h \circ f)$$

$h = B_2 u$.

<u>Fact</u>: $\Lambda^{-1} : \bar{U} \mapsto F'(\bar{U} \circ F)$ where $F = f^{-1}$.

$$\langle \Lambda u, v \rangle = \langle u, \Lambda^{-1} v \rangle.$$

<u>Transformation for Bergman kernel</u>:

$$K_1(z,w) = f'(z)\, K_2(f(z), f(w))\, \overline{f'(w)}$$

<u>Pf</u>: $\Lambda K_a = f'(K_a \circ f)$. $\qquad g \in H^2(\Omega_1)$.

$$\langle \underbrace{f'(K_a \circ f)}, g \rangle = \langle K_a, \Lambda^{-1} g \rangle$$

$$= \langle K_a, F'(g \circ F) \rangle$$

$$= \overline{\langle F'(g \circ F), K_a \rangle}$$

$$= \overline{F'(a)(g(F(a)))}$$

Hmmm. $\overline{F'(a)}\, K_{F(a)}$ has same effect on $g \in H^2(\Omega_1)$.

Conclude that $f'(K_a \circ f) = \overline{F'(a)}\, K_{F(a)}$

$$\boxed{f'(z)\, K_2(f(z), a) = K_1(z, F(a))\, \overline{F'(a)}}$$

Let $a = f(w)$ and use Chain Rule to get classic formula.

Can't resist: $f : \Omega \to D_1(0)$, $f_a(a) = 0$

$f_a'(a) > 0$

$\iint_{D_1(0)} h \, dA = \pi h(0)$. So $K_0 = \frac{1}{\pi}$ on $D_1(0)$

$\frac{1}{\pi} f' = f'(K_0 \circ f) = \overline{F'(0)}\, K_{F(0)} = c_1 K_a$

$$\boxed{f'(z) = c\, K(z,a)}$$

$c = \sqrt{\frac{\pi}{K(a,a)}}$

How to get: $c_1 = \overline{F'(0)} = \frac{1}{\pi f'(a)}$

>0

But $f'(a) = c\, K(a,a)$ from $\boxed{}$.

Solve for c.

Fun thing to do: $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$: $D_1(0) \circlearrowleft$

Know $K_0 \equiv \frac{1}{\pi}$. Use φ_a and trans. formula to compute

$$K(z,w) = \frac{1}{\pi(1-\bar{w}z)^2} \quad \text{on } D_1(0).$$

Facts: If Ω is simply connected and bounded with C^∞ boundary, then

$$K(z,w) = 4\pi\, S(z,w)^2$$

$$f_a'(z) = c\, K(z,a) \qquad c = \sqrt{\frac{\pi}{K(a,a)}}$$

$$f_a(z) = \frac{S(z,a)}{L(z,a)}$$