Thm: Ω bounded simply connected domain with C^∞ smooth boundary.

$$K(z,a) = 4\pi\, S(z,a)^2$$

Pf: $K_a = 4\pi\, S_a^2$

Know 1) $\overline{S_a} = \frac{1}{i} L_a T$ so $\overline{S_a}^2 = -L_a^2 T^2$

$\overline{S_a}^2\, \overline{T} = -L_a^2 T$

$$\boxed{S_a^2 T = -\overline{L_a}^2\, \overline{T}}$$

2) If $u \perp H^2(b\Omega)$ and $\overline{H^2(b\Omega)}$, then $u \equiv 0$.

Why: Given $\varphi \in C^\infty(b\Omega)$, solve D. Prob. Get $\varphi = h + \overline{H}$. Such a u would give $\langle u, \varphi \rangle_{b\Omega} = \cancel{\langle u, h\rangle}^{0} + \cancel{\langle u, \overline{H}\rangle}^{0} = 0$.

So $u \perp C^\infty(b\Omega)$. Oops. So $u \equiv 0$.

3) $H^2(b\Omega)^\perp = \{ \overline{h}\, \overline{T} : h \in H^2(b\Omega) \}$

$\overline{H^2(b\Omega)}^\perp = \{ h T : h \in H^2(b\Omega) \}$

Pf of Thm: Consider $\underbrace{(K_a - 4\pi S_a^2)T}_{v}$.

#3 says $v \perp \overline{H^2(b\Omega)}$.

Claim: $v \perp H^2(b\Omega)$ too. So $v \equiv 0$. ✓

Step 1: $\langle K_a T, h \rangle = \int_{b\Omega} K_a T \bar{h} \, ds$

$$= \int_{b\Omega} K_a \bar{h} \, dz \underset{\text{Green's}}{=} \iint_{\Omega} \frac{\partial}{\partial \bar{z}} (K_a \bar{h}) \, d\bar{z} \wedge dz$$

$$= \iint_{\Omega} K_a \overline{h'} \, \underbrace{d\bar{z} \wedge dz}_{2i \, dx \wedge dy}$$

$$= \langle 2i K_a, h' \rangle_{L^2(\Omega)} = \boxed{2i \overline{h'(a)}}$$

Note: Assume $h \in A^\infty(b\Omega)$.

Step 2: Claim: $\operatorname{Res}_a L_a^2 = 0$.

$$2\pi i \operatorname{Res}_a L_a^2 = \int_{b\Omega} L_a^2 \, dz = \int_{b\Omega} (L_a^2 T) \, ds$$

$$= -\int_{b\Omega} \left(\overline{S_a^2}\,\overline{T}\right) ds = -\overline{\int_{b\Omega} S_a^2\, dz}$$

$$= 0 \text{ by Cauchy's Thm.}$$

Conclude: $\displaystyle L_a(z)^2 = \frac{1}{4\pi^2 (z-a)^2} + H(z,a)$

where $H(z,a)$ analytic in z on Ω.

Step 3: $\langle K_a T, h\rangle = 2i\,\overline{h'(a)}$ from above.

Now $\displaystyle \langle 4\pi S_a^2 T, h\rangle = 4\pi \int_{b\Omega} S_a^2 T\, \overline{h}\, ds$

$$= -4\pi \int_{b\Omega} \overline{L_a}^2\, \overline{T}\, \overline{h}\, ds = -4\pi \int_{b\Omega} \overline{L_a}^2\, \overline{h}\, d\bar{z}$$

$$= -4\pi\, \overline{\int_{b\Omega} L_a^2\, h\, dz}$$

$$= -4\pi\, \overline{\left[2\pi i \operatorname{Res}_a L_a^2 h \right]}$$

$$= -4\pi \left[-2\pi i\, \frac{1}{4\pi^2}\, \overline{h'(a)} \right] = 2i\, \overline{h'(a)}.$$

Conclude that $(K_a - 4\pi S_a^2)T \perp A^\infty(b\Omega)$ in $L^2(b\Omega)$. So $\perp H^2(b\Omega)$ too by density. Done!

Domain with holes.

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \lambda_{ij} F_i'(z) \overline{F_j'(w)}$$

$F_i'(z) = 2\frac{\partial \omega_i}{\partial z}$ where ω_i is harmonic and $=1$ on γ_i boundry curve and $=0$ on others.

Quadrature Domains:

Unit disc:
$$\iint_{D_1(0)} h \, dA = \pi h(0)$$
Area quad identity

$$\int_{bD_1(0)} h \, ds = 2\pi h(0)$$
Boundary arc length quad identity.

Def: A bounded domain Ω is an area quadrature domain if there are points $\{z_n\}_{n=1}^N$ in Ω, and positive integers M_n, and complex constants c_{nm} such that

$$\iint_\Omega h\, dA = \sum_{n=1}^{N} \sum_{m=0}^{M_n} c_{nm}\, h^{(m)}(z_n)$$

for all $h \in H^2(\Omega)$.

Def^n: Call it a <u>simple</u> quad domain if $M_n = 0$ for each n. So $\iint h\, dA = \sum_{n=1}^{N} c_n h(z_n)$. $(*)$

<u>Lemma</u>: $(*)$ holds $\iff \sum_{n=1}^{N} \overline{c_n} K_{z_n} \equiv 1$

<u>Pf</u>: $\langle h, 1 \rangle = \iint h\, dA = \sum_1^N c_n h(z_n)$.

$$\langle h, \sum \overline{c_n} K_{z_n} \rangle = \sum_1^N c_n \langle h, K_{z_n} \rangle = \sum_1^N c_n h(z_n)$$

for all $h \in H^2(\Omega)$. Conclude $1 \equiv \sum_1^N \overline{c_n} K_{z_n}$.

<u>Thm</u>: Given bounded s.c. Ω with C^∞ smooth boundary, $\exists$ conformal mapping $f(z)$ as

close as desired in $C^\infty(\bar{\Omega})$ to z such that $f(\Omega)$ is a simple quadrature domain.

Pf: Know complex linear span of K_a as a ranges over Ω is dense in $A^\infty(\Omega)$.

So $\exists$ $\sum_1^N c_n K_{z_n}$ close to 1 in $A^\infty(\Omega)$.

Cook up f with $f' = \sum_1^N c_n K_{z_n} = \mathcal{K}$:

$$f(z) = a + \int_{\gamma_a^z} \underbrace{\mathcal{K}(w)}_{\approx 1}\, dw \approx z.$$

$\mathcal{K}$ close enough to 1 that $f : \Omega \xrightarrow[\text{onto}]{1\text{-}1} \tilde{\Omega}$.

Claim: $\tilde{\Omega}$ is a simple area quad. domain.

$$f' = \sum c_n K_{z_n} \qquad f : \Omega \to \tilde{\Omega}$$

$$F = f^{-1}$$

$$1 = \sum c_n \frac{1}{f'(z)} K(z, z_n)$$

$z = F(w)$

$$1 = \sum c_n F'(w) K_\Omega(F(w), z_n)$$

Trans.

$$1 = \sum_{n=1}^{N} c_n K_{\tilde{\Omega}}(w, F(z_n))$$ ← Formula

Aha! $\tilde{\Omega}$ is a quad domain:

$$\iint_{\tilde{\Omega}} h \, dA = \sum_{n=1}^{N} c_n h(F(z_n))$$

$$\forall h \in H^2(\tilde{\Omega}).$$