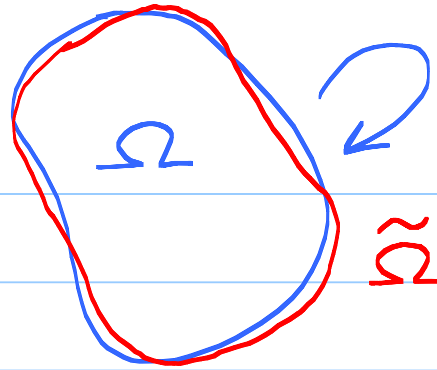
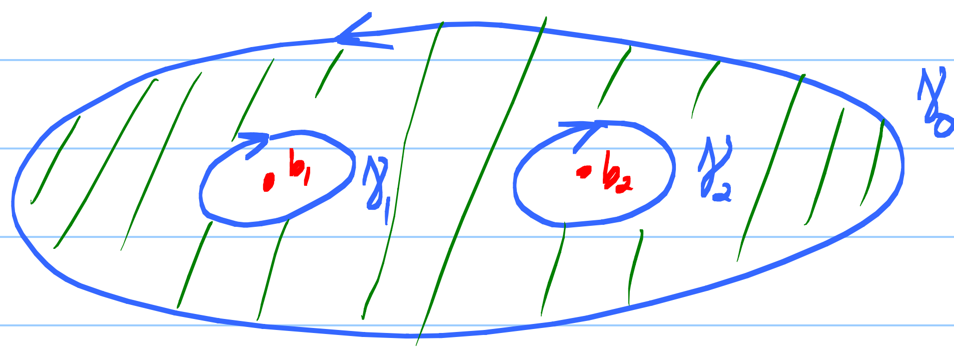


$f(z) \approx z$ $\qquad f' = \sum c_n K_{z_n} \approx 1$

$\tilde{\Omega} = f(\Omega)$ is a Q.D.

Proof correct for Ω simply connected last time.

Needed s.c. for
$$f(z) = a + \int_{\gamma_a^z} \sum c_n K_{z_n} \, dz$$
to get antiderivative f.



$\frac{1}{z-b_j}$ does not have an anlytic antiderivative on Ω.

<u>Easy fact</u>: Analytic h on Ω has an analytic antideriv. H $\iff$ $\int_\gamma h \, dz = 0$ for all closed γ in Ω $\iff$ $\int_{\gamma_j} h \, dz$ for $j = 1, 2, \ldots, n-1$.

From Ahlfors: $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ are a homology basis for Ω.

Def^n: γ homologous to zero in Ω if

$$\mathrm{Ind}_\gamma(a) = 0 \text{ for all } a \in \mathbb{C} - \Omega. \quad (\gamma \sim 0)$$

Fact: Given γ a cycle in Ω, $\exists$ integers $m_1, \ldots, m_{n-1}$ such that $\gamma \cup \left(\bigcup_{k=1}^{n-1} m_k \gamma_k \right) \sim 0$.

Gen^l Cauchy Thm: If $\Gamma \sim 0$, then $\int_\Gamma h\, dz = 0$ for analytic h.

How to cook up "antiderivatives" on multiply connected domains.

Given analytic h on Ω.

$$\int_{\gamma_j} h\, dz = w_j. \qquad \int_{\gamma_j} \frac{1}{z-b_j}\, dz = 2\pi i$$

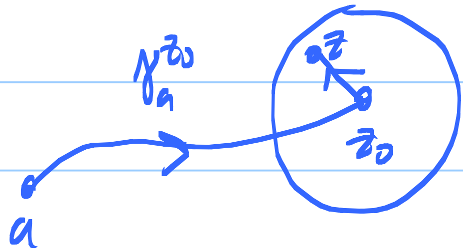
Idea: $g = h - \sum_{j=1}^{n-1} \frac{w_j}{2\pi i} \frac{1}{z-b_j}$ has zero periods around all γ_j's.

Let $$H(z) = \int_{\gamma_a^z} g(w)\, dw.$$

Easy to check

1) H is well defined (because $\int_{\text{closed}} g\, dw = 0$).

2) $H' = g$



<u>Thm</u>: Ω is bounded an n-connected domain, $b_j \in j$-th hole, $j = 1, \dots, n-1$.

Given analytic h on Ω, $\exists$ constants c_j and an analytic function H on Ω such that

$$H' = h + \sum_{j=1}^{n-1} \frac{c_j}{z - b_j}$$

<u>Back to the density of Q.D.s.</u>

1) Cook up $\varkappa_j = \sum c_n K_{z_n} \approx \dfrac{1}{z - b_j}$ in $C^\infty(\overline{\Omega})$.

Period matrix :

$$\lambda_{ij} = \int_{\gamma_i} \kappa_j \, dz \approx \int_{\gamma_i} \frac{1}{z-b_j} dz = 2\pi i \, \delta_{ij}$$

det $\neq 0$

By approximating close enough, can arrange that $\det[\lambda_{ij}] \neq 0$.

Step 1: Approximate 1 in Bergman span.

Get $h = \sum c_n K_{z_n} \approx 1$ in $C^\infty(\overline{\Omega})$.

Step 2: Since $h \approx 1$, the periods of h are close to zero! So there are small coefficients e_n so that

$$h - \sum e_n \kappa_n \quad \text{has zero periods.}$$

Why: $\int_{\gamma_i} h \, dz = e_n \int_{\gamma_i} \kappa_n \, dz = \lambda_{in} e_n$

small $\rightarrow$ $\left(\text{vector} \right) = \underbrace{[\lambda_{in}]}_{\det \approx (2\pi i)^{n-1}} \left(e_n \right) \leftarrow$ small too

Step 3: Get analytic f with

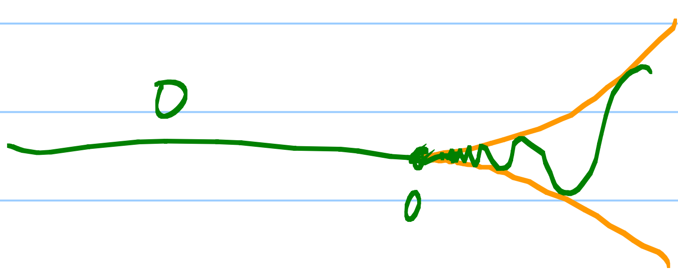
$$f' = \underbrace{h}_{\text{close to } 1} - \sum_{n=1}^{n-1} \underbrace{e_n}_{\text{close to zero}} x_n \longleftarrow \text{still close to } 1\,!$$

Finally f is close to z in $A^\infty(\Omega)$ and $f(\Omega)$ is a simple Q.D.

The case of real analytic boundaries:

γ: $z(t) = x(t) + iy(t)$, $a \le t \le b$, is a

real analytic curve if $x(t)$, $y(t)$ are real analytic, meaning they are given by convergent real power series $\sum_{n=0}^{\infty} a_n (t-\alpha)^n$. $C^\omega(a,b)$.



$e^{-1/t^2} \sin \frac{1}{t}$ $\quad$ C^∞ near 0. not real analytic.