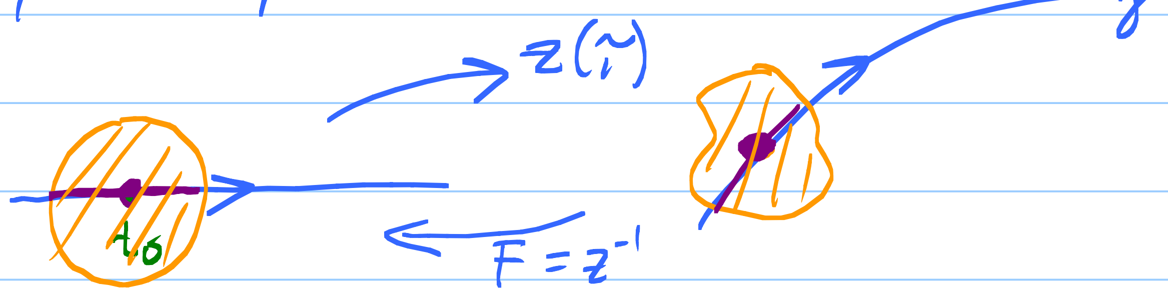
$z(t) = x(t) + i\,y(t)$ ← equal to real power series in t.

Matrix of periods $\int_{\gamma_j} \kappa_i \, dz$ non-singular

Fancier version: $\exists$ points a_i $i = 1, 2, \dots, n-1$ so that $\left[\int_{\gamma_j} K(z, a_i)\, dz \right]$ is non-singular.

Assume $z'(t) \neq 0$ for all t to get a <u>smooth</u> real analytic curve. Big idea: Replace t by a complex $\tilde{\tau}$.



Since $z'(t) \neq 0$, can shrink disc so $z(\tilde{\tau})$ is 1-1 on disc. Get biholo map.

<u>Observe</u>: A real analytic function on γ [meaning, any function λ on γ such that

$\lambda(z(t))$ is real analytic. Fact: does not depend on the choice of real analytic parametrization of γ], is the restriction of a holomorphic fcn locally to γ. Also the restriction of an anti-holo fcn!

Lemma: Given a real analytic function

$$h(x,y) = \sum_{n,m=0}^{\infty} c_{nm} x^n y^m = \sum_{n,m=0}^{\infty} a_{nm} z^n \bar{z}^m$$

in a nbhd of $\mathbb{R}$, there is a real analytic φ such that

1) $\varphi = 0$ on $\mathbb{R}$

2) $\frac{\partial \varphi}{\partial z} \equiv h$ on open set about $\mathbb{R}$.

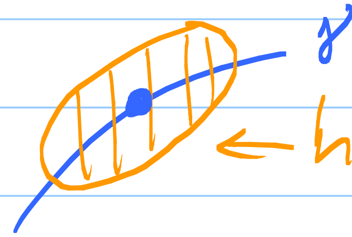
Pf: Clear that $\varphi = \sum_{n,m=0}^{\infty} \frac{a_{nm}}{n+1} z^{n+1} \bar{z}^m$ satisfies #2. Idea: subtract something ψ from φ with $\underbrace{\frac{\partial \psi}{\partial z} \equiv 0}_{\text{antianalytic}}$ to make #1 happen.

Aha! On $\mathbb{R}$, $\varphi = \sum \frac{a_{nm}}{n+1} x^{n+1} x^m$

on $\mathbb{R}$ too $\rightarrow$ $= \sum \frac{a_{nm}}{n+1} \bar{z}^{n+1} \bar{z}^m \leftarrow \psi$

Now $\varphi - \psi$ solves #1, #2.

<u>Lemma</u>: Lemma holds on real analytic curves γ: (h real analytic on nbhd of γ).



$\varphi|_\gamma = 0$

$\frac{\partial \varphi}{\partial z} \equiv h$ near γ.

<u>Lemma</u>: Given a domain Ω with real analytic boundary and real analytic h on an open set containing $b\Omega$, $\exists\ \varphi \in C^\infty(\bar{\Omega})$ such that $\varphi|_{b\Omega} = 0$ and $\frac{\partial \varphi}{\partial z} \equiv h$ on a nbhd of $b\Omega$.

<u>Pf</u>: Partition of unity plus previous lemma.

Cauchy Integral Formula

$\varphi(w) = 0$

h

$$u(z) - \varphi(z) = \underbrace{\frac{1}{2\pi i}\int_{b\Omega} \frac{u(w)}{w-z}\,dw}_{Cu} + \underbrace{\frac{1}{2\pi i}\iint_{\Omega} \frac{\frac{\partial u}{\partial w} - \frac{\partial \varphi}{\partial w}}{w-z}\,dw \wedge d\bar{w}}$$

$C_0^\infty(\Omega)$

analytic in z near $b\Omega$.

Conclude: If u is real analytic on $b\Omega$, then $\mathcal{C}u$ is real analytic on $b\Omega$. And, so $\mathcal{C}u$ extends to be analytic on an open set containing $\overline{\Omega}$. [Reason: real analytic fcn on $b\Omega$ is the restriction of an analytic fcn H to $b\Omega$ locally. But $H = \mathcal{C}u$ on open set of $b\Omega$. So $H \equiv \mathcal{C}u$ on Ω side, and H etends $\mathcal{C}u$ past $b\Omega$.]

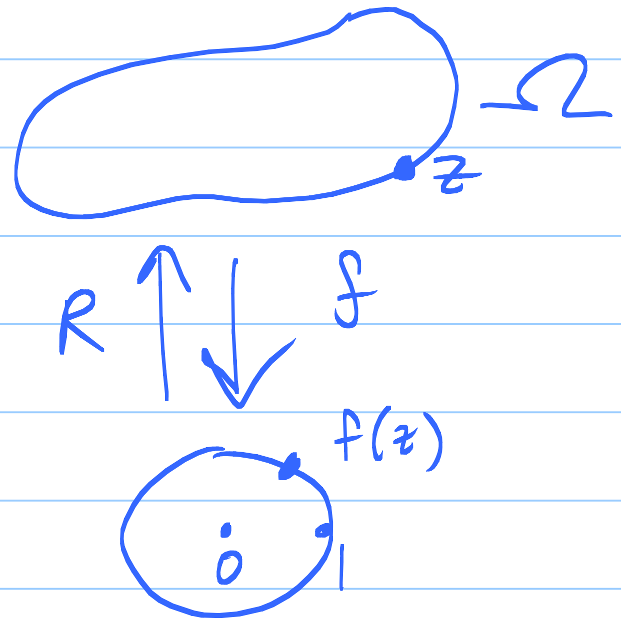
Remark: Local versions hold.

Thm: P maps $C^\omega(b\Omega)$ to analytic fcns on $\overline{\Omega}$. Local version holds.

Next time: Simply connected bounded area quadrature domains in $\mathbb{C}$ are exactly

Ω: $\exists$ Rational fcn $R(z)$ with poles outside $\overline{D_1(0)}$ which is 1-1 on $D_1(0)$ such that $\Omega = R(D_1(0))$.

How to get a Schwarz function $S(z)$ from R: $S(z) = \bar{z}$ on $b\Omega$ and $S(z)$ analytic on an open set containing $b\Omega$.



$R = f^{-1}$

$$R\Big(\underbrace{1/\overline{f(z)}}_{f(z) \text{ on } b\Omega}\Big) = z \quad \text{if } z \in b\Omega$$

Take conjugate: $$\underbrace{\overline{R\big(1/\overline{f(z)}\big)}}_{S(z)} = \bar{z}$$

Aha! S is analytic on $\overline{\Omega}$ except at finitely many poles inside Ω.

Fact: Any real analytic curve has a Schwarz fcn analytic on a nbhd of the curve. [Use the map to $\mathbb{R}$ above to cook up $S(z)$.]