<u>Thm</u>: Ω a bounded simply connected domain with smooth real analytic boundary. $C^\omega(b\Omega)$ denotes the space of smooth real analytic fcns on $b\Omega$.

1) Cauchy transform $\mathcal{C}: C^\omega(b\Omega)$

$: C^\omega(b\Omega) \to A(\overline{\Omega})$

2) Same for Szegö projection P.

3) If $\varphi \in C^\omega(b\Omega)$, then the solution to the D. Prob extends harmonically to an open set containing $\overline{\Omega}$.

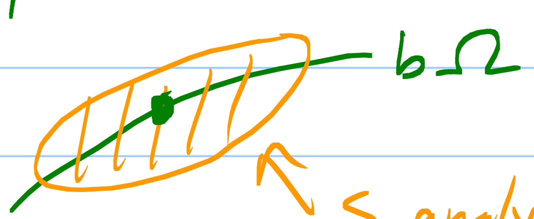
<u>Pf</u>: 1,2 done. 3: Sol^n to D. Prob is

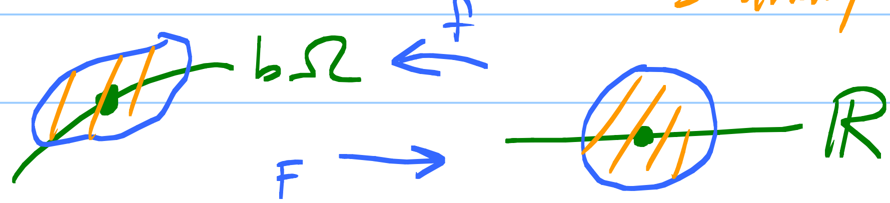
$$\frac{P(\varphi S_a)}{S_a} + \frac{\overline{P(\bar{\varphi} L_a)}}{\overline{L_a}}.\quad \text{Use (2).}$$

<u>Remark</u>: Kerzman-Stein kernel $A(z,w)$ is in $C^\omega(b\Omega \times b\Omega)$.

How to cook up a local Schwarz function:

$S(z) = \bar{z}$ on $b\Omega$





$F = f^{-1}$

f comes from $z(t) = x(t) + iy(t)$ power series.

$$f(\overline{F(z)}) = z \quad \text{on } b\Omega \quad \text{because}$$

$z \mapsto \bar{z}$ fixes $\mathbb{R}$. Now

$$S(z) = \overline{f(\overline{F(z)})} \quad \text{is analytic and}$$

$$S(z) = \bar{z} \quad \text{on } b\Omega.$$

<u>Another interesting thing:</u> $R(z) := f(\overline{F(z)})$

is an anti-holomorphic reflection function:

1) anti-holo near $b\Omega$,
2) $R(z) = z$ for $z \in b\Omega$.
3) R maps inside to the outside and vice versa.

Note: $R(z) = \overline{S(z)}$.

<u>Example:</u> Suppose Ω is a simple q.d., meaning

$$\iint_\Omega h \, dA = \sum_{j=1}^{N} c_j \, h(a_j).$$

Then $$1 = \sum_1^N \bar{c}_j K_{a_j}. \quad (*)$$

Let $f : \Omega \to D_1(0)$ be a Riemann map and $F = f^{-1}$. Know Transf. Formula

$$f'(z)\, K_{D_1}(f(z), w) = K_\Omega(z, F(w))\, \overline{F'(w)}$$

or $$K_{D_1}(w, f(z))\, \overline{f'(z)} = F'(w)\, K_\Omega(F(w), z)$$

Apply this to $(*)$:

$$F'(1 \circ F) = \sum_1^N \bar{c}_j\, F'(z)\, K_\Omega(F(z), a_j)$$

$$= \sum_1^N \bar{c}_j\, K_{D_1}(z, \underbrace{f(a_j)}_{w_j})\, \overline{f'(a_j)}$$

$$F' = \sum_{j=1}^N b_j \frac{1}{\pi(1 - \bar{w}_j z)^2}$$

Finitely many double poles outside $\overline{D}_1$ residue free.

So F = Rational with finitely many simple poles outside $\overline{D}_1$.

So Ω = Rational image of D_1.

Same thing works for general q.d.:

$$\iint_\Omega h \, dA = \sum_{j=1}^N \sum_{m=0}^{n_j} c_j^m\, h^{(m)}(a_j)$$

$$f'(z)\, K_{D_1}(f(z), w) = K_{\Omega}(z, F(w))\, \overline{F'(w)}$$

diff'tiate w.r.t. $\bar{w}$.

Quad Ident: $\Longleftrightarrow$ $1 = \sum_j c_j^{(m)} K_{a_j}^{(m)}$

where $K_{a_j}^{(m)} = \frac{\partial^m}{\partial \bar{w}^m} K(\cdot, w) \Big|_{w=a_j}$

Note: $\langle h, K_{a_j}^{(m)} \rangle = h^{(m)}(a_j)$.

Theorem: Ω is a bounded simply connected area quadrature domain $\Longleftrightarrow$ there is a rational function $R(z)$ with poles outside $\overline{D_1(0)}$ such 1) R is 1-1 on $D_1(0)$, and

2) $R(D_1(0)) = \Omega$.

Pf: $\Rightarrow$ done. $\Leftarrow$. First cook up Schwarz function

$f: \Omega \to D_1(0)$

$\Omega \leftarrow D_1(0) : R = f^{-1}$.

$$S(z) = \underbrace{\overline{R\left(1/\overline{f(z)}\right)}}_{\substack{\text{analytic} \\ \text{on nbhd of } b\Omega.}} = \bar{z} \quad \text{on } b\Omega$$

<u>Note</u>: $S(z)$ is meromorphic on $\overline{\Omega}$ with no poles on $b\Omega$.

<u>Claim</u>: This property yields quad. identity.

$$\iint_\Omega h \, dA = \iint_\Omega h \left(\frac{-1}{2i} \, dz \wedge d\bar{z} \right)$$

$$= \frac{1}{2i} \iint_\Omega \underbrace{h(z)}_{\frac{\partial}{\partial \bar{z}}\left(h(z)\bar{z}\right)} \, d\bar{z} \wedge dz$$

$$\underset{\text{Green's}}{=} \frac{1}{2i} \int_{b\Omega} h \underbrace{\bar{z}}_{=S(z)} \, dz = \frac{1}{2i} \int_{b\Omega} h \, S(z) \, dz$$

$$= 2\pi i \sum_{\substack{\text{inside} \\ \Omega}} \operatorname{Res} hS = \text{linear combo of values of } h \text{ and derivatives.}$$

Shown: $S(z)$ extends mero to $\Omega \implies$ Ω is a g.d.

Thm: Ω is a s.c. q.d. $\iff \Omega = R(\mathbb{D}, (0))$ $\iff$ S extends mero to Ω.

Gustafsson: Multiply connected Ω is a q.d. $\iff \Omega = R(\tilde{\Omega})$ where R extends meromorphically to the double of $\tilde{\Omega}$.