More about $K_a^{(m)}$.

$$\left[\begin{array}{l} \varphi_a(z) = \varphi(z-a) \quad \text{where} \quad \varphi(z) = \frac{1}{\varepsilon^2}\varphi_0(z/\varepsilon), \\ \varphi_0 \in C_0^\infty(D_1(0)) \text{ real, } \geq 0, \text{ radially symmetric, and} \\ \iint \varphi_0 \, dA = 1 \end{array}\right.$$

$$h(a) = \iint_\Omega h \, \varphi_a \, dA = \langle h, \varphi_a \rangle = \langle h, \underbrace{B\varphi_a}_{K_a} \rangle$$

$$h^{(m)}(a) = \iint_\Omega \frac{\partial^m h}{\partial z^m} \varphi_a \, dA = (-1)^m \iint_\Omega h \frac{\partial^m \varphi_a}{\partial z^m} \, dA$$

$$= \langle h, \Psi_a^m \rangle = \langle h, \underbrace{B\Psi_a^m}_{K_a^{(m)}} \rangle$$

where $\Psi_a^m = (-1)^m \overline{\dfrac{\partial^m \varphi_a}{\partial z^m}}$

$$= (-1)^m \frac{\partial^m \varphi_a}{\partial \bar{z}^m}$$

Fact: $K_a(z) = K(z,a)$ is analytic in z and anti-analytic in a. $\quad K_a^{(m)}(z) = \dfrac{\partial^m}{\partial \bar{w}^m} K(z,w)\Big|_{w=a}$.

Step 1: Show $K_a(z)$ is anti-analytic in a:

$$K_a(z) = (B\varphi_a)(z)$$

$$\frac{\partial}{\partial a} K_a(z) = B\left(\frac{\partial \varphi_a}{\partial a}\right)(z) = \underbrace{\int K(z,w)\,\frac{\partial \varphi_a}{\partial a}(w)\,dA}_{= \langle -\frac{\partial \varphi_a}{\partial w}, K_z \rangle = 0}$$

$= \langle \frac{\partial \varphi_a}{\partial a}, K_z \rangle$

But $\varphi_a(w) = \varphi(w-a)$.

So $\frac{\partial}{\partial a}\varphi_a(w) = -\frac{\partial \varphi}{\partial w}(w-a)$ orthogonal to $H(\Omega)$.

So $K_a(z)$ is anti-analytic in a.

<u>Step 2:</u> Easy to see $K_a^{(m)}(z) = \frac{\partial^m}{\partial \bar{w}^m} K(z,w)\Big|_{w=a}$.

<u>Back to general bounded simply connected q.d.</u>

$$\iint_\Omega h\,dA = \sum_{j=1}^{N} \sum_{m=0}^{n_j} c_{jm}\, h^{(m)}(a_j)$$

$$\langle h, 1 \rangle = \langle h, \sum \bar{c}_{jm} K_{a_j}^{(m)} \rangle$$

for all h. So $\boxed{1 = \sum \bar{c}_{jm} K_{a_j}^{(m)}}$

Next, we want to use transf. formula for the Bergman kernels and the R. Map. fcn

$f : \Omega \to D_1(0) \qquad F = f^{-1}.$

$$F'(z)\, K_\Omega(\underset{z \in D_1}{F(z)}, \underset{w \in \Omega}{w}) = K_{D_1}(z, f(w))\, \overline{f'(w)}$$

Take $\frac{\partial^m}{\partial \bar{w}^m}$ of this formula:

$$F' \cdot (K_w^{(m)} \circ F) = \frac{\partial^m}{\partial \bar{w}^m} K_{D_1}(z, f(w))\, \overline{f'(w)}$$

$$\boxed{1 = \sum \bar{c}_{jm} K_{a_j}^{(m)}}$$

multiply by F' and compose with F

$$F' = \sum \bar{c}_{jm} F' \cdot (K_{a_j}^{(m)} \circ F) = \text{Linear combo of}$$

$$\frac{\partial^{(m)}}{\partial \bar{w}^m} K_{D_1}(z, w) \Big|_{w = a_j}$$

<u>Note</u>: $K_{D_1}(z,w) = \frac{1}{\pi} \frac{1}{(1-\bar{w}z)^2}$

$$K_{D_1}^{(1)} = \frac{2z}{\pi(1-\bar{w}z)^3}$$

Hmmm. $K_0^{(m)} = c z^m$ on disc.

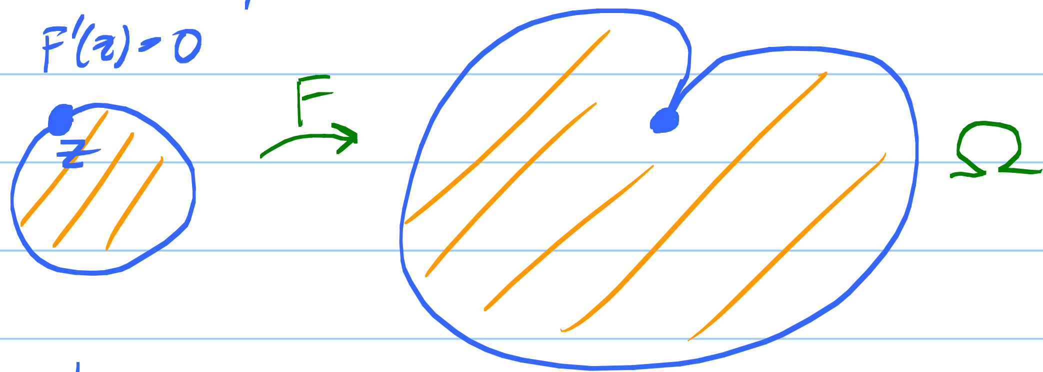
$$K_{D_1}^{(n)} = \frac{(n+1)!\, z^n}{\pi(1-\bar{w}z)^{n+2}}$$

All residue free.

We've shown F' is rational with poles outside $D_1(0)$, residue free. So F is rational. $\Omega = F(D_1(0))$.

Remark: Know F is 1-1. So $F' \neq 0$ on $D_1(0)$.

If $F'(z) = 0$ for $z \in bD_1(0)$, we get a cusp in boundary of the q.d.



$F'(z)$ can only have a simple zero.

Same theory for the Szegő kernel and boundary arc length q.d. works.

Transformation Rule for Szegő kernels.

$$f : \Omega_1 \to \Omega_2 \quad \text{biholo}, \quad F = f^{-1}$$

$$\sqrt{f'(z)}\, S_2(f(z), w) = S_1(z, F(w))\, \sqrt{\overline{F'(w)}}$$

$\lambda\, \varphi = \sqrt{f'}\cdot \varphi \circ f$ is similar to Λ.

Pf: In simply connected case, so it is easy to see $\sqrt{f'}$ is single valued analytic on Ω_1.

$$\langle h, \sqrt{f'}\,(S_a \circ f) \rangle = \int_{b\Omega} h \, \overline{\sqrt{f'}} \, \overline{(S_a \circ f)} \, ds$$