Claim: "Easy" to see that

$$K_a^{(m)}(z) = \frac{\partial^m}{\partial \bar{w}^m} K(z,w)\Big|_{w=a}$$

$h'(a) = \langle h, K_a^{(1)} \rangle$ $\qquad h(a) = \iint_{w\in\Omega} K(a,w)h(w)\,dA$

$$\left[K(z,w) = \langle K_w, K_z \rangle = \overline{\langle K_z, K_w \rangle} = \overline{K(w,z)} \right]$$

Idea: To diff'tiate under $\iint$, assume $h = B\Psi$ where $\Psi \in C_0^\infty(\Omega)$.

$$h(a) = (B\Psi)(a) = \iint_\Omega K(a,w)\Psi(w)\,dA$$

$$h'(a) = \iint_\Omega \frac{\partial}{\partial a} K(a,w)\,\Psi(w)\,dA$$

$=$ for all $\Psi \in C_0^\infty$.

$\rightarrow$ $h'(a) = \langle h, K_a^{(1)} \rangle = \langle \Psi, K_a^{(1)} \rangle$

So $K_a^{(1)} = \overline{\frac{\partial}{\partial a} K(a,w)} = \frac{\partial}{\partial \bar{a}} K(w,a)$. ✓

Transformation formula for Szegő kernels. Assume C^∞ smooth boundaries. $f: \Omega_1 \xrightarrow[\text{biholo}]{} \Omega_2$.

will show $f \in C^\infty(\bar{\Omega}_1)$. This implies f' is in $C^\infty(\bar{\Omega}_1)$ and f' is non-vanishing on $\bar{\Omega}_1$. Same for $F = f^{-1}$.

Fact: A single valued branch of $\sqrt{f'}$ exists.

Note: $\sqrt{f'}$ is in $C^\infty(\bar{\Omega}_1)$, too.

Def^n: $\lambda(\varphi) = \sqrt{f'} \cdot (\varphi \circ f)$. Thm: λ is an isometry $L^2(b\Omega_2) \longrightarrow L^2(b\Omega_1)$.

$$H^2(b\Omega_2) \to H^2(b\Omega_1).$$

Pf: Parametrize $b\Omega_1$: $z_1(t)$

$$b\Omega_2: \quad z_2(t) = f(z_1(t)).$$

$$z_2'(t) = f'(z_1(t))\, z_1'(t).$$

$$ds_2 = |z_2'(t)|\, dt = |f'(z_1(t))| \underbrace{|z_1'(t)|\, dt}_{ds_1}$$

$$\|\sqrt{f'}(\varphi \circ f)\|^2_{L^2(b\Omega_1)} = \int_{b\Omega_1} |f'|\, |\varphi \circ f|^2\, ds_1$$

$$= \int |f'(z_1(t))|\, \Big|\varphi\big(\underbrace{f(z_1(t))}_{z_2(t)}\big)\Big|^2 |z_1'(t)|\, dt$$

$$= \int |\varphi(z_2(t))|^2\, |z_2'(t)|\, dt$$

$$= \int |\varphi|^2\, ds_2$$

<u>Next</u>: $\langle \lambda(u), \lambda(v) \rangle = \langle u, v \rangle$

$$\int \sqrt{f'}\,(u \circ f)\, \overline{\sqrt{f'}}\, \overline{(v \circ f)}\, ds_1$$

$$= \int |f'|\, (u \bar{v}) \circ f\, ds_1$$

$$= \int u \bar{v}\, ds_2 \quad \text{as above.}$$

<u>Finally</u>: $\lambda(\varphi) = \sqrt{f'}\,(\varphi \circ f)$.

$$\lambda^{-1}(\Psi) = \sqrt{F'}\,(\Psi \circ F).$$

<u>Why</u>: $F'(z) = \dfrac{1}{f'(F(z))}$

Take $\sqrt{F'}$ to be $= \dfrac{1}{\sqrt{f'(F(z))}}$ ← branch picked above

Get λ^{-1} via inverse function formula.

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$$\langle \lambda u, v \rangle = \langle u, \lambda^{-1} v \rangle$$

$v = \lambda \bar{V}$ $\quad \langle \lambda u, \lambda \bar{V} \rangle = \langle u, \bar{V} \rangle$ ✓

Transf Rule for Szegő kernel:

$$\sqrt{f'(z)}\, S_2(f(z), w) = S_1(z, F(w))\, \overline{\sqrt{F'(w)}}$$

Pf: $\langle h, \sqrt{f'}\, S_w(f) \rangle = \langle \lambda^{-1} h, S_w \rangle$

$$= (\lambda^{-1} h)(w)$$

$$= \sqrt{F'(w)}\, h(F(w))$$

Aha! $\langle h, \overline{\sqrt{F'(w)}}\, S_{F(w)} \rangle$ has same effect!

const

So $\sqrt{f'}\, S_w \circ f = \overline{\sqrt{F'(w)}}\, S_{F(w)}$. ✓

Same thing:

$$S_1(z, w) = \sqrt{f'(z)}\, S_2(f(z), f(w))\, \overline{\sqrt{f'(w)}}$$

Pf: Let $w = f(\bar{W})$ above and use

inversion formula.

Can use this to show, given a C^∞ smooth simply connected quad domain with respect to arc length Ω and Riemann map f,

$$1 = \sum c_j^{(m)} S_{a_j}^{(m)}$$

compose with $F = f^{-1}$ and multiply by $\sqrt{F'}$.

$$\sqrt{F'} = \sum c_j^{(m)} \sqrt{F'} \cdot S_{a_j}^{(m)} \circ F$$

$=$ linear combo of

$$\left.\frac{1}{2\pi i(1-\bar{w}z)}\right|_{w=a_j}$$

and derivatives in w.

Conclude $\sqrt{F'}$ is <u>rational</u>!