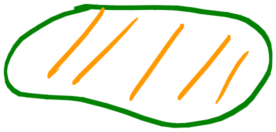
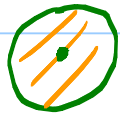
 Ω simply connected, bounded

$F = f^{-1} \uparrow \downarrow f$, Riemann Map

 $D_1(0)$

Bergman Span of Ω = all complex linear combos of $K_a^{(m)}$, $a \in \Omega$, $m = 0, 1, 2, \ldots$

Szegő Span: same with S_a in place of K_a.

<u>Theorems</u>: 1) Ω is an area QD $\Longleftrightarrow$

F' is in the Berg. Span of $D_1(0)$

$\Longleftrightarrow$ F' is rational <u>and</u> residue free, no poles on $\overline{D_1(0)}$,

$\Longleftrightarrow$ F is rational, no poles on $\overline{D_1(0)}$.

2) Ω is C^∞ smooth too. Ω is a boundary arc-length QD $\Longleftrightarrow$

$\sqrt{F'}$ is in the Szegő span of $D_1(0)$

$\Longleftrightarrow$ $\sqrt{F'}$ is rational, no poles in $\overline{D_1(0)}$.

<u>How to cook up double QDs</u>: Start with an Ω as above with C^∞ smooth bndry. We know $F \in C^\infty(\overline{D_1(0)})$ and F' non-vanishing. So $\sqrt{F'} \in A^\infty(D_1(0))$ too. Can approximate

$\sqrt{F'}$ by polynomial $P(z)$ in $A^\infty(D, (0))$.

[$h(rz) \approx$ power series, $r<1$ close to 1.]

Plan: Cook up $\tilde{F}$ so that $\sqrt{\tilde{F}'} = P(z)$.

Easy! $\tilde{F}$ = anti-derivative of $P(z)^2$.

Thm above yields that $\tilde{\Omega} = \tilde{F}(D, (0))$ is a QD wrt area and arc-length, i.e., a double QD.

Remark: $\tilde{\Omega}$ is close to Ω and it is a one point double QD, meaning

$$\iint_{\tilde{\Omega}} h\, dA = \sum_{m=0}^{N} c_m h^{(m)}(a)$$

one point $a = \tilde{F}(0)$

$$\int_{b\Omega} h\, ds = \text{same thing, different } c_m\text{'s.}$$

Good things about double QDs

Area QD: a) Schwarz function $S(z)$ extends meromorphically to Ω. [$S(z) = \bar{z}$ on $b\Omega$]

b) Bergman kernel: $K(z,w) = R(z, S(z), \bar{w}, \overline{S(w)})$ where R is rational!

(Similar to unit disc where $S(z) = 1/z$)

c) Riemann map f is a rational combo of z and $S(z)$. Since $S(z) = \bar{z}$ on $b\Omega$, see that $f = R(z, \bar{z})$ on $b\Omega$.

d) $S(z)$ is algebraic: $\exists$ poly $P(z,w)$ such that $P(z, S(z)) \equiv 0$.

Arc Length QD: A) $T(z)$ is rational in z and $\bar{z}$.

B) $\exists$ fcns g and G mero on Ω with finitely many poles on Ω that extend continuously up to $b\Omega$ such that

$$T(z) = g(z) \text{ on } b\Omega, \text{ \underline{and} }$$

$$T(z) = \overline{G(z)} \text{ on } b\Omega.$$

C) The Kerzman-Stein kernel $A(z,w)$ is rational in $z, \bar{z}, w, \bar{w}$.

D) $\mathcal{C}$ and P send rational fcns on $b\Omega$ to same.

<u>Easy Theorem</u>: Suppose $f: \Omega_1 \to \Omega_2$ is biholo map between bounded domains with C^∞ smooth boundary. Then $f \in C^\infty(\overline{\Omega_1})$.

Pf:

$$\begin{tikzcd} \Omega_1 \arrow[r, "f"] \arrow[d, leftrightarrow] & \Omega_2 \arrow[d, leftrightarrow] \\ \tilde{\Omega}_1 \arrow[r, "\tilde{f}"] & \tilde{\Omega}_2 \end{tikzcd}$$

smooth maps

area QDs

$$1 = \sum{}' c_{nm} K_{a_n}^{(m)}$$

compose with $\tilde{f}$ and multiply by $\tilde{f}'$ and use transf formula.

Get $\tilde{f}'$ = element of the Berg. Span.

So $\tilde{f}'$ is in $C^\infty(\bar{\tilde{\Omega}}_1)$. Pull back to Ω_1 to see $f \in C^\infty(\bar{\Omega}_1)$.

Repeat for $F = f^{-1}$. Easy to see f' non-vanish on $b\Omega_1$ since $F' = 1/f' \circ F$.

<u>Green's Function:</u> on D, (0): Think w fixed, $|w|<1$.

$$G(z,w) = -\ln\left|\frac{z-w}{1-\bar{w}z}\right|.$$

$$= -\ln|z-w| + \underbrace{\left(\text{harmonic in } z\right)}_{\text{vanishing in } z \text{ on } |z|=1}$$