Orthonormal Bases:

Berg Space $H^2(\Omega)$ $\{h_n(z)\}$ $\perp$ basis, o.n.

$$K_a = \sum c_n h_n \quad \text{where } c_n = \langle K_a, h_n \rangle$$

$$= \overline{\langle h_n, K_a \rangle} = \overline{h_n(a)}$$

$$K(z,a) = K_a(z) = \sum_{n=1}^{\infty} \overline{h_n(a)}\, h_n(z)$$

Same for S_a in $H^2(b\Omega)$.

<u>Unit disc:</u> z^n are orthogonal in $H^2(D_1(0))$ <u>and</u> $H^2(bD_1(0))$.

$$\langle z^n, z^n \rangle_{L^2(D_1(0))} = \int_{D_1(0)} |z|^{2n}\, dA = \int_0^{2\pi}\int_0^1 r^{2n}\, r\,dr\,d\theta$$

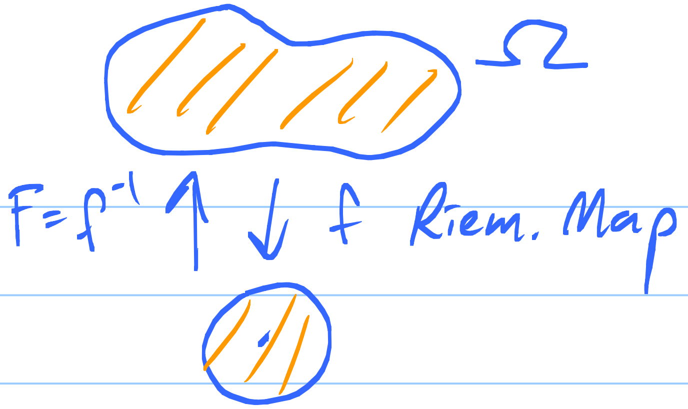
$$= 2\pi \frac{1}{2n+2} r^{2n+2}\Big|_0^1$$

$$= \frac{\pi}{n+1}, \quad \sqrt{\frac{n+1}{\pi}}\, z^n \text{ o.n.}$$

$$\langle z^n, z^n \rangle_{L^2(bD_1(0))} = 2\pi, \quad \frac{z^n}{\sqrt{2\pi}} \text{ o.n.}$$

<u>Thm</u>: Isoperimetric inequality is trivial!

Suppose Ω is a bounded simply conn. domain bounded by a Jordan real analytic curve.



Know f and F extend analytically past boundaries.

Pf: (T. Carleman) Trick: expand $\sqrt{F'}$ in power series and compare norms.

$$\sqrt{F'} = \sum_{n=0}^{\infty} a_n z^n$$

Step 1: $A(\Omega) = \iint_{D_1(0)} |F'|^2 \, dA$ because

$$|F'|^2 = |u_x + i v_x|^2 = u_x^2 + v_x^2 = \leftarrow \text{yes!}$$

$$\text{Jacobian det} = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \underset{\uparrow \, CR}{=} \det \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} =$$

So $A(\Omega) = \left\langle \left(\sqrt{F'}\right)^2, \left(\sqrt{F'}\right)^2 \right\rangle$

But
$$\left(\sqrt{F'}\right)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) z^n$$
$$= \sum_{n=0}^{\infty} \sqrt{\frac{\pi}{(n+1)}} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) \sqrt{\frac{(n+1)}{\pi}} \, z^n$$

Now $A(\Omega) = \sum_{n=0}^{\infty} \frac{\pi}{(n+1)} \left| \sum_{k=0}^{n} a_k a_{n-k} \right|^2$

Use Cauchy-Schwarz : $\left| \sum_{k=0}^{n} b_k \right|^2 = \left| (b_0, \ldots, b_n) \cdot (1,1,1,\ldots 1) \right|^2$

$$\leq \left(\sum |b_k|^2 \right) \cdot (n+1)$$

$$\underset{\uparrow}{A}(\Omega) = \leq \underbrace{\pi \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k|^2 |a_{n-k}|^2}_{\pi \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^2}$$

area of Ω

<u>Next</u> : $\underset{\uparrow}{L}(\Omega) = \int_{bD_1(0)} |F'| \, ds$ $\qquad z(t) = e^{it}$

length $b\Omega$

$F(z(t))$

$L(\Omega) = \int_0^{2\pi} |F'(z(t))| \, |z'(t)| dt$

$$= \langle \sqrt{F'}, \sqrt{F'} \rangle_{bD_1(0)}$$

But $\sqrt{F'} = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \left(\sqrt{2\pi}\, a_n \right) \frac{z^n}{\sqrt{2\pi}}$.

So $L(\Omega) = \sum_{n=0}^{\infty} 2\pi |a_n|^2 = 2\pi \sum_{n=0}^{\infty} |a_n|^2$.

Conclude $\boxed{A(\Omega) \leq \frac{1}{4\pi} L(\Omega)^2.}$

What if we get equality. Cauchy-Schwarz ineq vectors must line up. Get

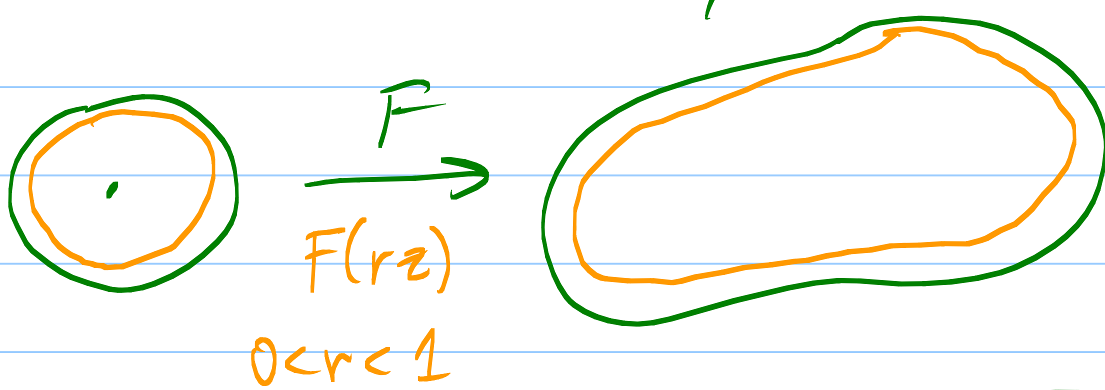
$$a_n = \lambda^n a_0, \quad \text{and} \quad f(z) = \frac{1}{1-\lambda z}.$$

Ω not a half-plane. So Ω is a disc.

$$A = \pi r^2, \quad L = 2\pi r. \quad r = \frac{L}{2\pi}$$

$$A = \pi \left(\frac{L}{2\pi}\right)^2 \checkmark$$

What to do if $b\Omega$ is only C^∞ smooth.



Let $r \nearrow 1$ and use $F \in C^\infty(\overline{D_1(0)})$.

Interesting things:

$$A(\Omega) = \iint |F'|^2 \, dA = \frac{1}{2i} \iint F' \, \overline{F'} \, dz \wedge d\bar{z}$$

$$= \frac{1}{2i} \iint \frac{\partial}{\partial z} \left(F \, \overline{F'}\right) dz \wedge d\bar{z}$$

$$\underset{\text{Green's}}{=} \quad \frac{1}{2i}\int_{bD_1(0)} F\,\overline{F'}\,d\bar{z} = \frac{1}{2i}\int_{b\Omega} w\,d\bar{w}$$