Dragan Vukotić, The isoperimetric inequality and a theorem of Hardy and Littlewood, Math. Monthly 110, 2003.

Area Q.D. $S(z)=\bar{z}$ on $b\Omega$, and $S(z)$ extends meromorphically to Ω.

Arc Length Q.D. Suppose Ω is C^∞ smooth boundary and bounded. Ω is an arc length QD. $\Longleftrightarrow$ $T(z)$ extends meromorphically to Ω. $T(z)=h\big|_{b\Omega}$ ← mero on Ω, finitely many poles in Ω, and h extends C^∞ smoothly to $b\Omega$.

Notice: $T(z(t))=\dfrac{z'(t)}{|z'(t)|}$ has unit modulus. So

$$T(z)=1/\overline{T(z)} = 1/\overline{h(z)}. \quad \leftarrow \boxed{\bar{T}=\frac{1}{h}}$$

Pf: $(\Longleftarrow)$. $$\int_{b\Omega} g\, ds = \int_{b\Omega} g \underbrace{\bar{T}}_{\frac{1}{h}} \underbrace{T\, ds}_{dz}$$

$$= \int_{b\Omega} g\cdot\frac{1}{h}\, dz = 2\pi i \sum_{\substack{\text{zeroes}\\ \text{of } h \text{ in } \Omega}} \operatorname{Res} \frac{g}{h}$$

$$= \sum c_{nm}\, g^{(m)}(a_n)$$

$(\Rightarrow)$. $\langle g, S_a \rangle = g(a)$

$$\langle g, S_a^{(m)} \rangle = g^{(m)}(a)$$

$S(z,a) = S_a(z)$. Similar to $K_a^{(m)}$ story,

can show $S_a^{(m)}(z) = \frac{\partial^m}{\partial \bar{a}^m} S(z,a)$.

Think $g(a) = \int_{z \in b\Omega} S(a,z)\, g(z)\, ds$

$$g^{(m)}(a) = \int_{z \in b\Omega} \frac{\partial^m}{\partial a^m} S(a,z)\, g(z)\, ds$$

$$= \langle g, \underbrace{\frac{\partial^m}{\partial \bar{a}^m} S(\cdot,a)}_{S_a^{(m)}} \rangle$$

Ω is an arc-length QD $\Leftrightarrow$

$\langle g, \underline{\underline{1}} \rangle = \int g\, ds = \sum c_{nm}\, g^{(m)}(a_n) \quad \forall\, g \in H^2(b\Omega)$

$$= \langle g, \underbrace{\sum \bar{c}_{nm} S_{a_n}^{(m)}}_{\text{must } = 1.} \rangle$$

Basic identity: $\overline{S_a} = \frac{1}{i} L_a T$

$$\overline{S(z,a)} = \frac{1}{i} L(z,a)\, T(z)$$

Differentiate in a-variable:

$$\overline{\frac{\partial^m}{\partial \bar{a}^m} S(z,a)} = \frac{1}{i} \frac{\partial^m}{\partial a^m} L(z,a)\, T(z)$$

$$\boxed{\overline{S_a^{(m)}} = \frac{1}{i} L_a^{(m)}\, T} \leftarrow \text{on } b\Omega.$$

Ω arc-length OD:

$$1 \equiv \sum c_{nm} \overline{S_{a_n}^{(m)}} = \underbrace{\frac{1}{i}\left(\sum c_{nm} L_a^{(m)}\right)}_{\mathcal{L}(z)} T$$

Aha! $\quad T(z) = 1/\mathcal{L}(z) \leftarrow$ mero!

__Remark:__ $T(z)$ and $S(z)$ are related.

$$S(z) = \bar{z} \quad \text{on } b\Omega.$$

$z(t)$ parametrizes boundary.

$$S(z(t)) = \overline{z(t)} \leftarrow$$

diff w.r.t. t and use complex chain rule

$$\frac{\partial S}{\partial z}\frac{dz}{dt} + \underbrace{\cancel{\frac{\partial S}{\partial \bar{z}}\frac{d\bar{z}}{dt}}}_{0} = \underbrace{\cancel{\frac{\partial \bar{z}}{\partial z}\frac{dz}{dt}}}_{0} + \frac{\partial \bar{z}}{\partial \bar{z}}\frac{d\bar{z}}{dt}$$

$$S'(z(t))\, z'(t) = \overline{z'(t)} \quad \leftarrow \text{divide by } |z'(t)|$$

$$\boxed{S'(z)\, T(z) = \overline{T(z)}}$$

Hmmm. $S'(z) = \overline{T(z)}^{\,2}$ If Ω is an arc-length QD, then $\overline{T(z)} = h(z) \leftarrow$ mero.

and $S' = h^2 \leftarrow$ no simple poles!

Fun facts about $D_1(0)$. Can solve the Dirichlet problem with rational data using only algebra.

Rational: $R(x,y)$ $\quad z = x+iy$

$$R\left(\frac{z+\bar z}{2}, \frac{z-\bar z}{2i}\right) = \tilde R(z, \bar z)$$

Green's Function: $G(z,w) = \mathrm{Ln}\left|\frac{z-w}{1-\bar w z}\right|$

$$G(z,w) = \underbrace{\mathrm{Ln}|z-w| + H_w(z)}_{\text{zero in } z \text{ on } b\Omega.} \leftarrow \text{harmonic in } z, \text{ removable sing at } w.$$

Think $w \in \Omega$ fixed.

Calculation:

$$\frac{1}{2}\frac{\partial}{\partial w}\ln\left|\frac{z-w}{1-\bar{w}z}\right|^2=\frac{1}{2}\frac{\partial}{\partial w}\ln\left(\frac{z-w}{1-\bar{w}z}\right)\overline{\left(\frac{z-w}{1-\bar{w}z}\right)}$$

$$=\frac{1}{2}\frac{\partial}{\partial w}\ln\frac{z-w}{1-w\bar{z}}\cdot\frac{\bar{z}-\bar{w}}{1-\bar{w}z}$$

$$=\frac{1}{2}\frac{1}{\frac{z-w}{1-w\bar{z}}\cdot\frac{\bar{z}-\bar{w}}{1-\bar{w}z}}\left[\frac{\partial}{\partial w}\left(\frac{z-w}{1-w\bar{z}}\right)\right]\left(\frac{\bar{z}-\bar{w}}{1-\bar{w}z}\right)$$

$$=\frac{1}{2}\cdot\frac{\frac{-(1-|z|^2)}{(1-\bar{z}w)^2}}{\frac{z-w}{1-w\bar{z}}}=\frac{-(1-|z|^2)}{2(z-w)(1-\bar{z}w)}$$

$$=\underbrace{\frac{1}{2}\frac{1}{z-w}+u_w(z)}_{\text{zero for } z\in b\Omega.}\longleftarrow \text{harmonic in } z,\ \text{removable sing at } w.$$

Big idea: $R(z,\bar{z})=R(z,\tfrac{1}{z})$

$\nwarrow s(z)=\frac{1}{z}$

$$R(z,\tfrac{1}{z})-\sum c_n\frac{\partial^n}{\partial w^n}G(z,w_j)\longleftarrow \text{cancel off poles of } R(z,\tfrac{1}{z})$$