On $D_1(0)$: $\quad G(z,w) = \operatorname{Ln}\left|\frac{z-w}{1-\bar{w}z}\right| = \underbrace{\operatorname{Ln}|z-w| + u(z,w)}_{=0 \text{ in } z \text{ on } b\Omega}$ $\quad w\in\Omega$

$$\frac{\partial}{\partial w} G(z,w) = \underbrace{\frac{-(1-|z|^2)}{2(z-w)(1-w\bar{z})}}_{\text{rational in } w,\, z,\, \bar{z}} = \underbrace{-\frac{1}{2}\cdot\frac{1}{(z-w)} + \frac{\partial u}{\partial w}(z,w)}_{=0 \text{ in } z \text{ on } b\Omega}$$

$$\frac{\partial^m}{\partial w^m} G(z,w) = \frac{-(m-1)!}{2(z-w)^m} + \text{harmonic in } z$$

rational in $w, z, \bar{z}$ too.

How to solve D. Prob. on $D_1(0)$ with rational bndry data $R(z,\bar{z})$.

Trick: Schwarz function $S(z) = \frac{1}{z}$ on $D_1(0)$

Now $H(z) = R(z, \frac{1}{z})$ is meromorphic on $\overline{D_1(0)}$ without poles on $|z|=1$. Can subtract off principal parts at all poles inside $D_1(0)$ using

$$\frac{\partial^m}{\partial w^m} G(z, a_j), \leftarrow \text{zero on } z\in b\Omega.$$

Now $R(z,\frac{1}{z}) - \sum c_{jm} \frac{\partial^m}{\partial w^m} G(z,a_j)$

has removable sing inside $D_1(0)$, has $R(z,\bar{z})$ for boundary values. So it is the sol^n to the D. Prob. It is rational in z

and $\bar{z}$! <u>Fact</u>: Can pull apart rational harmonic fcns on $D_1(0)$ as $R(z) + Q(\bar{z})$ where R, Q rational.

<u>Thm</u>: (B, Ebenfeld, ~~Gustafsson~~, Khavinson, Schapiro) If Ω satisfies the rational-to-rational property like the unit disc, then Ω <u>is</u> a disc.

Better way to calculate $\frac{\partial}{\partial w} G(z,w)$.

<u>First</u>: $\frac{\partial}{\partial w} \operatorname{Ln}|w| = \frac{\partial}{\partial w}\left(\frac{1}{2} \operatorname{Ln}|w|^2\right)$

$$= \frac{1}{2}\frac{\partial}{\partial w} \operatorname{Ln}(w\bar{w}) = \frac{1}{2} \frac{1}{w\bar{w}} \cdot \underbrace{\frac{\partial}{\partial w}(w\bar{w})}_{\bar{w}} = \frac{1}{2}\cdot\frac{1}{w}$$

<u>Second</u>: Next use chain rule again to compute

$$\frac{\partial}{\partial w} \operatorname{Ln}\Big|\underbrace{\varphi_w(z)}_{\text{Möbius}}\Big|$$

<u>Consequence</u>: Given $\varphi \in C(\{|z|=1\})$, approx φ by rational functions on $\{|z|=1\}$, $R_n(z,\bar{z})$. Now get sol^{n}s u_n for $R_n(z,\bar{z})$. Max Princ shows $U = \operatorname{Lim} u_n$ solves D. Prob.

for data φ.

What about Q.D.s? $\quad f : \Omega_1 \to \Omega_2$

$$G_1(z,w) = G_2(f(z), f(w)).$$

$f : \Omega \to D_1(0)$ a Riemann map.

$$G(z,w) = \operatorname{Ln}\left| \frac{f(z)-f(w)}{1-\overline{f(w)}\,f(z)} \right|$$

Can repeat above: $\quad R(z,\bar{z})$.

$$R(z, \underbrace{S(z)}_{\substack{\uparrow \\ \text{algebraic} \\ P(z,S(z))\equiv 0 \\ \uparrow \text{poly}}}) - \sum c_{jm} \underbrace{\frac{\partial^m}{\partial w^m} G(z,w)}_{\text{rational combo's of } f(z),\ \overline{f(z)},\ f(w) \text{ and its derivatives}}$$

Fact: f is algebraic. So derivatives are too.

Conclude sol^n to D. Prob is real algebraic.

Famous conjecture: If sol^n on Ω to the D. Prob with polyonomial data is always a poly, then Ω has to be a disc or an ellipse!

Ex: $D_1(0)$. $P(x,y) = P\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = p(z,\bar{z})$

$$= \sum c_{nm} z^n \bar{z}^m$$

Case $n > m$: $z^n \bar{z}^m = z^{n-m} \underbrace{|z\bar{z}|^m}_{1}$ on $b\Omega$

etc: $p(z,\bar{z}) = \underbrace{(\text{poly in } z) + (\text{poly in } \bar{z})}_{\text{extends to } D_1(0) \text{ as harmonic fcn that solves the D. Prob.}}$ on $b\Omega$

Ex: The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$.

Trick: Fischer's Map. $q(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

$$u \overset{\mathcal{J}}{\longmapsto} \underbrace{\Delta\big(\underbrace{q\,\underset{\deg N}{u}}_{\deg(N+2)}\big)}_{\deg N.}$$

Linear trans of vector space Polys($\deg \leq N$).

Claim: $\mathcal{J}$ is 1-1.

Why: $\Delta(qu) \equiv 0 \implies qu$ harmonic.

But $q = 0$ on bEllipse. So Max Princ

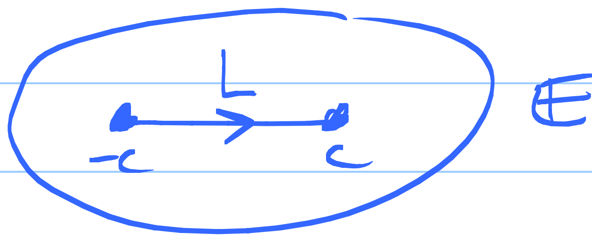
$\implies qu \equiv 0. \implies u \equiv 0.$

Conclude: $\tilde{\Delta}$ is onto!

Now, given poly boundary data $p(z,\bar z)$, get a poly sol^n to $\Delta(qu) = \Delta p(z,\bar z)$.

Now $p(z,\bar z) - qu$ gives poly sol^n to D. Prob.!

Ellipse as a Q.D.:



$$\iint_E h\, dA = \int_L h\, \rho\, ds$$

weight fcn on L