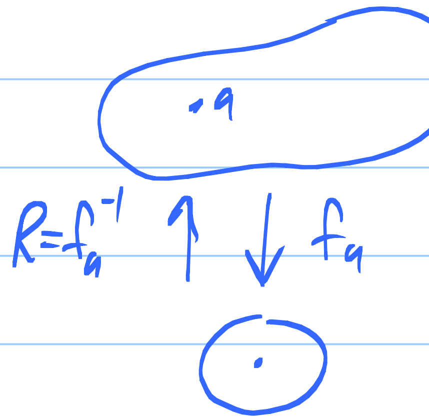
Results about D. Prob from last time are for simply connected Ω area QD.



Ω is an area Q.D. simp. conn.

Riemann map

Know $R = f_a^{-1}$ is rational.

Consequently $z = R(f_a(z))$

So $f_a = R^{-1}$ is algebraic.

Also $S(z) = \overline{f_a^{-1}\left(1/\overline{f_a(z)}\right)} = \overline{R\left(1/\overline{f_a(z)}\right)}$

is seen to be algebraic too!

Recall $R(z,\bar z) = R(z, S(z)) - \sum c_{jm} \underbrace{\frac{\partial^m}{\partial w^m} G(z, a_j)}_{=0 \text{ on } b\Omega}$

$$G(z,w) = \operatorname{Ln}\left|\frac{f_a(z) - f_a(w)}{1 - \overline{f_a(w)}\, f_a(z)}\right|$$

and $\frac{\partial^m}{\partial w^m} G(z,w)$ is rational in $f_a(z)$ and $\overline{f_a(z)}$ for fixed w.

Remark: Solution to D. Prob. with data $R(z,\bar{z})$ rational is an algebraic function that is a rational fcn of $f_a(z)$ and $\overline{f_a(z)}$.

Classical Neumann Problem for Δ:

Suppose Ω is a bounded simp. conn. domain with C^∞-smooth bndry. Given $\psi \in C^\infty(b\Omega)$, find harmonic u on Ω such that

$$\frac{\partial u}{\partial n} = \psi \text{ on } b\Omega.$$

Remark: If u exists, then

$$\int_{b\Omega} \frac{\partial u}{\partial n}\, ds \underset{\text{Gauß}}{=} \iint_{\Omega} \underbrace{\Delta u}_{=0}\, dA = 0$$

Must have $\displaystyle\int_{b\Omega} \psi\, ds = 0$.

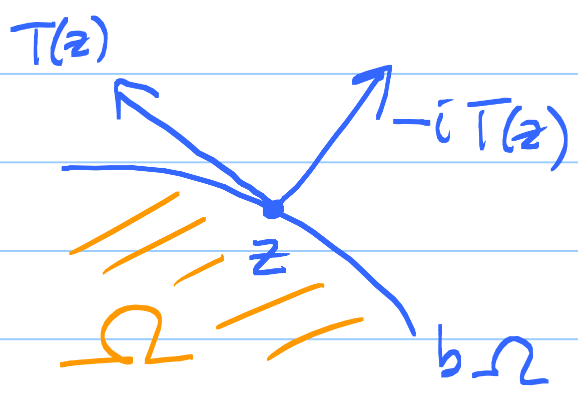
Big idea: If u solves problem, write

$u = h + \overline{H}$ where h, H analytic.

<u>Fact</u>: $\tau \in \mathbb{C}$, $|\tau| = 1$. $z(t) = z_0 + t\tau$

$$\frac{d}{dt} h(z(t))\Big|_{t=0} = h'(z(t))\, z'(t)$$

$$= h'(z_0)\, \tau$$



$$\boxed{\frac{\partial h}{\partial n}(z) = -i\, h'(z)\, T(z)}$$

$$\frac{\partial y}{\partial n} = -i h' T + i \overline{H'}\, \overline{T} \underset{\substack{\uparrow \\ \text{want}}}{=} \psi$$

Hmmm. $\overline{S_a} = \frac{1}{i} L_a T$ so $T = \dfrac{i \overline{S_a}}{L_a}$

$$-i h' \left(\frac{i \overline{S_a}}{L_a} \right) + i \overline{H'}\, \overline{T} \underset{\substack{\uparrow \\ \text{want}}}{=} \psi$$

Divide by $\overline{S_a}$:

$$\underbrace{\frac{h'}{L_a} + i \frac{\overline{H'}}{\overline{S_a}}\, \overline{T}}_{\text{orthog sum!}} = \frac{\psi}{\overline{S_a}}$$

See $\dfrac{h'}{L_a} = P(\Psi/\bar{S}_a)$

$$\boxed{h' = L_a P(\Psi/\bar{S}_a)}$$

Also, get $\boxed{H' = S_a P(\bar{\Psi}/\bar{L}_a)}$

(*)

<u>Note</u>: $P(\Psi/\bar{S}_a)(a)$ ← Cancels pole of L_a if $=0$.

$$= \int_{z \in b\Omega} \underbrace{S(a,z)}_{\overline{S_a(z)}} \Psi(z)/\overline{S_a(z)}\, ds$$

$$= \int \Psi\, ds = 0. \quad \text{Yes!}$$

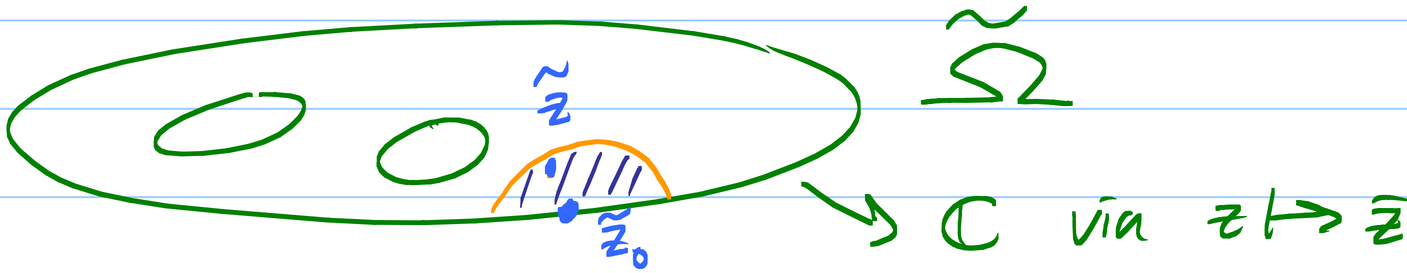
<u>Thm</u>: Given $\Psi \in C^\infty(b\Omega)$ with $\int_{b\Omega} \Psi ds = 0$, the sol^n to the N. Prob. with data Ψ is

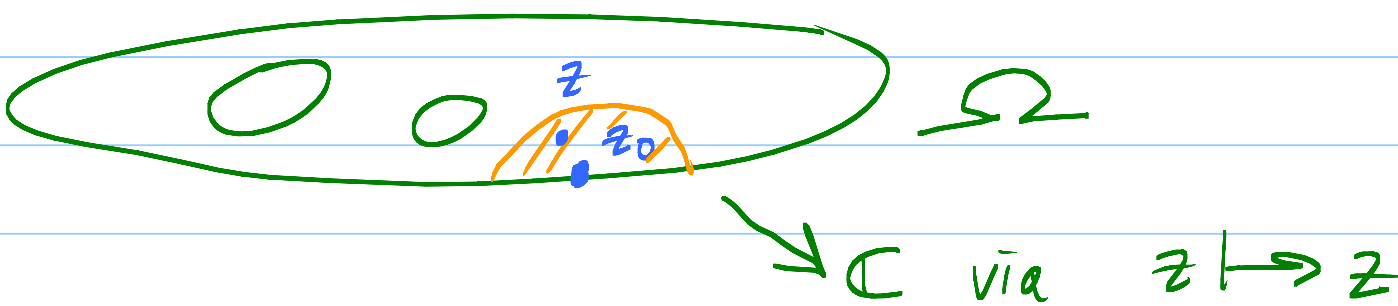
$u = h + \bar{H}$ where h and H are analytic antiderivatives given from (*) above.

Conclude that $u \in C^\infty(\bar{\Omega})$.

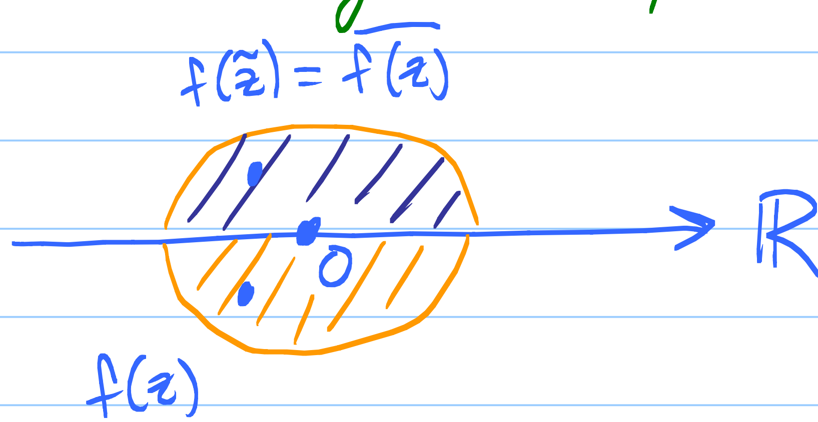
Note: Sol^n u is unique up to addition of a const. (See from Green's Identities.)

The double of Ω, a bounded n-connected C^∞-smooth domain.



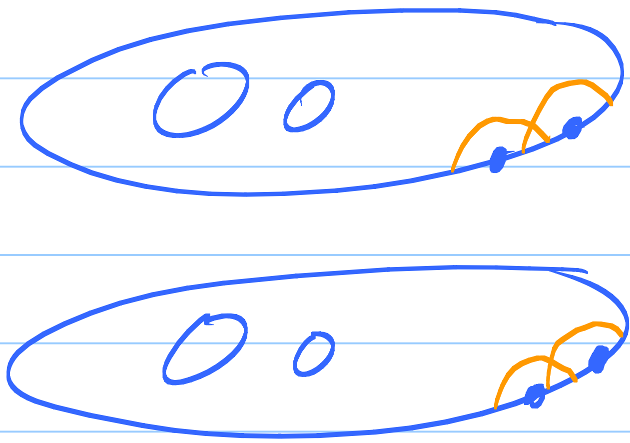


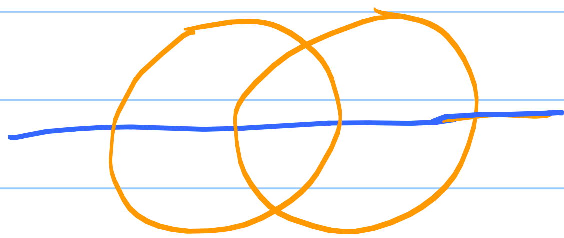
Glue Ω to $\tilde{\Omega}$ along boundary like this.



[Note: Fill in holes, use Riemann map to $D_1(0)$, map to lower half plane via an LFT and restrict to small nbhd of z_0. For inside hole, invert via $1/(z-w_0)$ and then do this.]

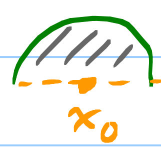
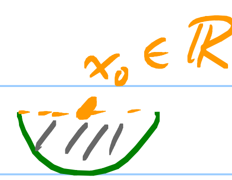
In this way, we get a Riemann surface.



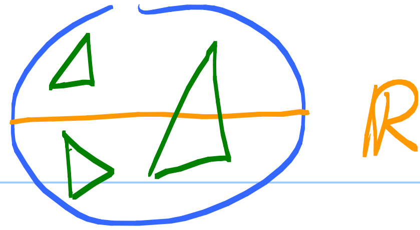


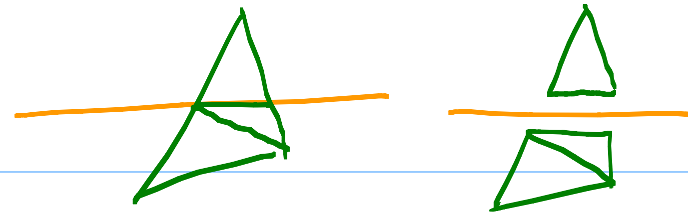
Change of variables is locally biholo by Schwarz refl!

Genus $g = n-1$.

Thm: If f is continuous on $D_r(x_0)$ and analytic on  and  , then f is analytic on $D_r(x_0)$.

Pf: Morera's plus Cauchy-Goursat.

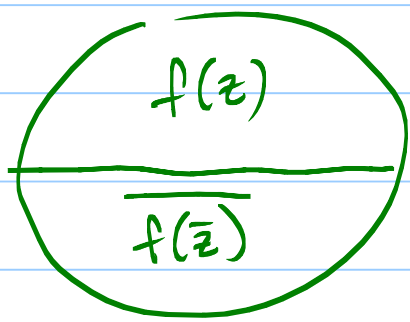




ε-slide back.

<u>Remark</u>: Schwarz reflection follows.

$f(z)$ real along $\mathbb{R}$ means



piece together and are continuous past $\mathbb{R}$.