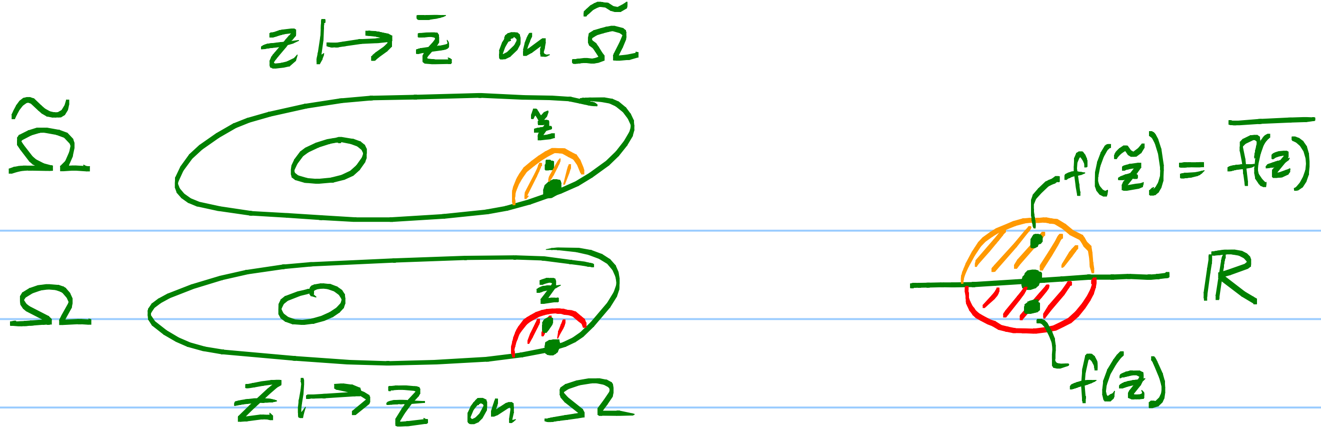


"Double of Ω" $= \Omega \cup b\Omega \cup \tilde{\Omega}$ is a compact Riemann Surface $\hat{\Omega}$.

Meromorphic functions on $\hat{\Omega}$.

<u>Fact</u>: f is mero on $\hat{\Omega}$ if and only if

$$f = \begin{cases} h & \text{on } \Omega \text{ where } h \text{ mero on } \Omega \\ \bar{H} & \text{on } \tilde{\Omega} \text{ where } H \text{ mero on } \Omega \approx \tilde{\Omega} \end{cases}$$

and $h = \bar{H}$ on $b\Omega$ away from finitely many possible poles on $b\Omega$.

<u>Ex</u>: Suppose f_a is a Riemann map $\Omega \to D_1(0)$, Ω is simply connected. Since $|f_a| = 1$ on $b\Omega$, we have $f_a(z) = 1/\overline{f_a(z)}$ on $b\Omega$. This gives a meromorphic extension of f_a to $\hat{\Omega}$.

Also note that $1/f_a(z) = \overline{f_a(z)}$ on $b\Omega$.

And $1/f_a(z)^n = \overline{f_a(z)^n}$ on $b\Omega$

Hmmm. Get a mero fcn on $\hat{\Omega}$ with a single pole of order n at $a \in \Omega$.

<u>Fact</u>: Given a mero fcn G on $\hat{\Omega}$ without poles on $b\Omega$, then G is a rational fcn of extension of f_a to $\hat{\Omega}$.

Note: $f_b(z) = c\,\dfrac{f_a(z) - f_a(b)}{1 - \overline{f_a(b)}\, f_a(z)}$

Subtract off powers of $f_a(z)$ and $1/f_a(z)$ to remove poles of G in Ω and $\tilde{\Omega}$.

Get $G - \sum^{N} c_n f_a^n - \sum^{M} b_m f_a^{-m}$ (on Ω side)

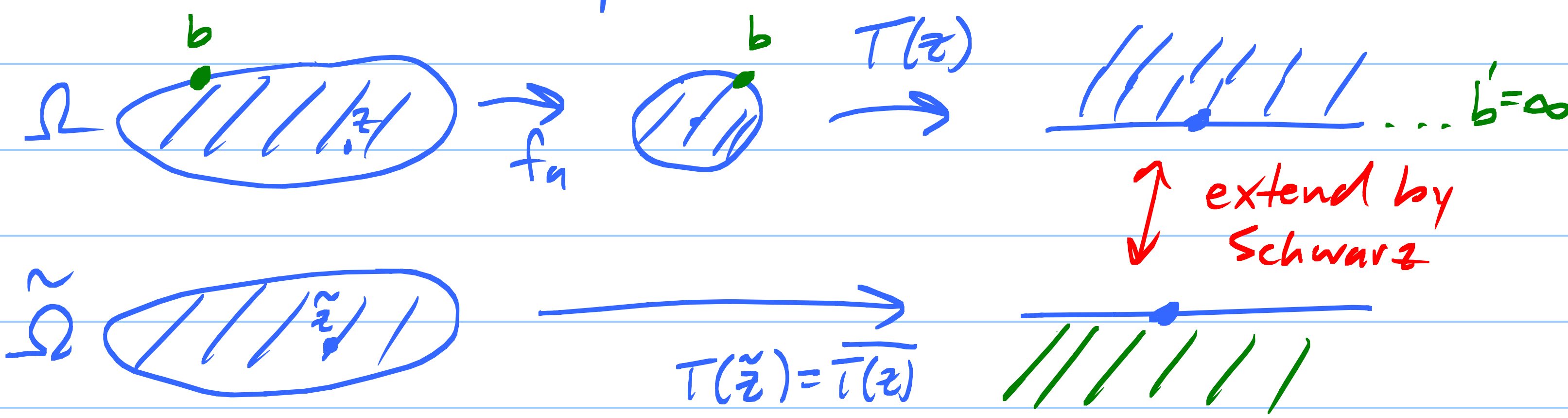
$G - \sum c_n \frac{1}{\bar{f}_a^n} - \sum b_m \bar{f}_a^m$ (on $\tilde{\Omega}$ side)

without poles at a or $\tilde{a}$. Repeat for other poles b using f_b as above. Get an analytic fcn on a compact Riemann Surface. Max Princ $\Rightarrow$ it is constant.

Conclude:

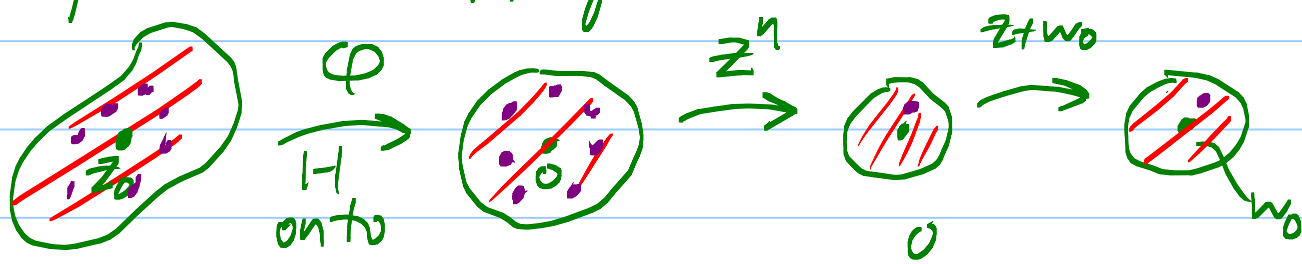
<u>Thm</u>: The field $\mathcal{M}$ of mero fcns on the double of a s.c Ω consists of all rational combo's of the extension of a single R. map to $\hat{\Omega}$.

How to deal with poles on $b\Omega$:



<u>Property of meromorphic fcns</u>: Given $G \in \mathcal{M}$, $\exists$ a positive integer m called the degree of G such that, given any point $w \in \hat{\mathbb{C}}$, $\exists$ m points $z_1, z_2, \dots, z_m$ in $\hat{\Omega}$ such that $G(z_j) = w$ counted with multiplicity. At all but finitely many points $w \in \hat{\mathbb{C}}$, the z_j's are m distinct points.

<u>Why</u>: Local mapping thm.



Get a locally constant total multiplicity fcn on a connected set. It must be const.

EX: On an area quadrature domain Ω, we know $\exists$ mero Schwarz fcn $S(z)$ with $S(z) = \bar{z}$ on $b\Omega$. Aha! z extends meromorphically to $\hat{\Omega}$ via

$$g \begin{cases} z & \text{on } \Omega \\ \overline{S(\tilde{z})} & \text{on } \tilde{\Omega} \end{cases}$$

Hmmm. So does $S(z)$ via

$$h \begin{cases} S(z) & \text{on } \Omega \\ \bar{\tilde{z}} & \text{on } \tilde{\Omega}. \end{cases}$$

Fact: These two form a "primitive pair" for $\mathcal{M}$, meaning that all fcns in $\mathcal{M}$ are given as rational combos of the two.

Condition: $\exists\, w \in \mathbb{C}$ such that $h^{-1}(w)$ consists of exactly m points of multiplicity one and g separates those m points (meaning

$$g(z_j) \neq g(z_k) \quad \text{if} \quad j \neq k, \quad h^{-1}(w) = \{z_1, z_2, \ldots, z_m\}.$$

Show that g, h form a primitive pair.

$S(z)$ has finitely many poles in Ω and $|z| < M$ in Ω. Pick a $w \in \mathbb{C}$ with w "close" to ∞ ($|w| > M$) so that $h^{-1}(w)$ separates into n points with mult one near a pole of order n. Get $\{z_1, \dots, z_m\} = h^{-1}(w)$ distinct points in Ω with h-multiplicity one. But $g(z) = z$ on the Ω side. So of course g seperates the distinct z_j.