Ω an area QD.

$$h = \begin{cases} S(z) & \text{on } \Omega \\ \bar{z} & \text{on } \tilde{\Omega} \end{cases}$$

$$g = \begin{cases} z & \text{on } \Omega \\ \overline{S(z)} & \text{on } \tilde{\Omega} \end{cases}$$

Step 1: Got a w_0 close to ∞ so that

$$h^{-1}(w_0) = \{z_1(w_0), z_2(w_0), \ldots, z_m(w_0)\}$$

consists of m distinct points of mult 1.

Step 2: If we specify $|w_0| > \max_{z \in \bar{\Omega}} |z|$, then

all the $z_j(w_0)$ are in Ω and $g(z)$ separates them, e.g., $\underset{g(z_j(w_0))}{\underset{\shortparallel}{z_j(w_0)}} \neq \underset{g(z_k(w_0))}{\underset{\shortparallel}{z_k(w_0)}}$ if $j \neq k$. on Ω

Step 3: Given f mero on $\hat{\Omega}$, do this

$$h^{-1}(w) = z_1(w), \ldots, z_m(w) \quad \text{counted}$$

with mult.. (Note: get m distinct points except at possibly finitely many $w \in \hat{\mathbb{C}}$.)

Want: $$f(z_j(w)) = \sum_{n=1}^{m} R_n(\underset{h(z_j(w))}{\underset{\shortparallel}{w}})\, g(z_j(w))^n, \quad j = 1, \ldots, m$$

Cramer's Rule:

$$R_n(w) = \frac{\det \bar{W}_n}{\det \bar{W}}$$

does not depend on order of $z_j(w)$'s

where $\bar{W} = \left[g(z_j(w))^k \right]$

and $\bar{W}_n = \begin{pmatrix} \text{same thing with } f(z_j(w)) \\ \text{in place of } n\text{-th column} \end{pmatrix}$

R_n is mero near w_0 (and denom $\neq 0$ there). R_n is analytic except at possibly finitely many points. Easy to see via the local mapping thm that R_n can only blow up to finite order at isolated singularities in $\mathbb{C}$. So they are poles or removable. Same for $w = \infty$. Conclude each $R_n(w)$ is <u>rational</u>.

$$f(z_j(w)) = \sum_{n=1}^{m} R_n\Big(\underbrace{h(z_j(w))}_{w} \Big)\, g(z_j(w))^n$$

Every z in $\hat{\Omega}$ gets hit by some $z_j(w)$ as w ranges over $\hat{\mathbb{C}}$. So

$$f(z) = \sum_{n=1}^{m} R_n(h(z))\, g(z)^n$$

and we see that f is a rational combo of g and h.

[True fact on general compact Riemann Surfaces. See Farkas and Kra. But must show a mero fcn exists, and cook up a primitive pair.]

Consequence: If Ω simply connected QD

$f_a = R^{-1} \rightarrow$ $z = R(f_a)$, R rational

$S(z) = Q(f_a)$, Q rational

$f_a = r(z, S(z))$, r rational.

Ha! $f_a(z) = r(z, \bar{z})$ for $z \in b\Omega$.

Green's Function and the Bergman kernel.

$$K(z,w) = -\frac{2}{\pi} \frac{\partial^2 G}{\partial z \partial \bar{w}}(z,w)$$

Recall $G(z,w) = -\ln|z-w| + u(z,w)$

$= -\ln|z-w| + \left(\begin{array}{c}\text{sol}^n \text{ to D. prob} \\ \text{with data } \ln|z-w|\end{array}\right)$

$$G(\cdot, w)\big|_{b\Omega} = 0 \quad \text{for } w \in \Omega.$$

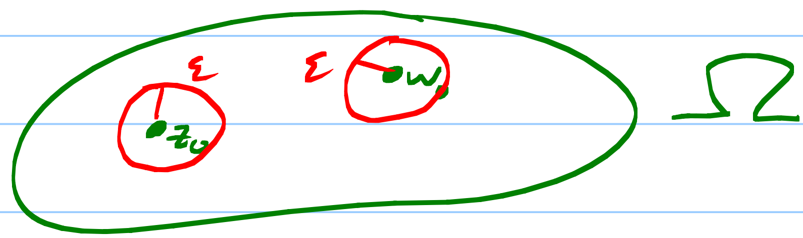
<u>Facts</u>: 1) $G(z,w) = G(w,z)$

2) $G_z(z,w) = -\dfrac{1}{2(z-w)} + u_z(z,w)$

3) $G_{z\bar{w}}(z,w) = 0 + u_{z\bar{w}}(z,w)$

<u>Note</u>: Can differentiate in w and $\bar{w}$ under the operator that solves the D. prob.

<u>Pf of (1)</u>:



$$\Omega_\varepsilon = \Omega - \left(D_\varepsilon(z_0) \cup D_\varepsilon(w_0)\right)$$

$$u = G_{z_0} = G(\zeta, z_0)$$

$$v = G_{w_0} = G(\zeta, w_0)$$

Green's identity: $0 = \displaystyle\iint_{\Omega_\varepsilon} u\underbrace{\Delta v}_{0} - v\underbrace{\Delta u}_{0}\, dA$

$$= \int_{b\Omega_\varepsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$$

zero on $b\Omega$

$$= -\int_{bD_\varepsilon(z_0)} - \int_{bD_\varepsilon(w_0)}$$

Consider $\displaystyle\int_{bD_\varepsilon(w_0)} u \frac{\partial v}{\partial n}\, ds = \int_0^{2\pi} G_{z_0} \underbrace{\frac{\partial}{\partial r}\left(-\ln r + u\right)}_{\left(\frac{-1}{r} + \text{Bdd}\right)} \underbrace{ds}_{r\,d\theta}$

where $r = \varepsilon$.

$\rightarrow G_{z_0}(w_0) = G(w_0, z_0)$ as $\varepsilon \to 0$.

Other one $\rightarrow -G(z_0, w_0)$. ✓

Pf that $K(z,w) = -\frac{2}{\pi} \frac{\partial^2 G}{\partial z \partial \bar{w}}(z,w)$

$h(z_0) = \langle h, K_{z_0} \rangle$ characterizes K_{z_0}.

analytic

Another important kernel:

$$\Lambda(z,w) = -\frac{2}{\pi} \frac{\partial^2 G}{\partial z \partial w}(z,w)$$

$$= -\frac{1}{\pi} \frac{1}{(z-w)^2} + u_{zw}(z,w)$$

Λ is analytic in z, w. Double pole at w.