$$G(z,w_0) = -\ln|z-w_0| + u(z,w_0)$$
$$= -\ln|z-w_0| + \begin{pmatrix} \text{sol}^n \text{ to D. Prob.} \\ \text{with boundary} \\ \text{data } \ln|z-w_0| \end{pmatrix}$$

Harmonic in z, zero in z on $b\Omega$, singularity $-\ln|z-w_0|$ at w_0.

Regularity of D. Prob show that

$G_a(z) = G(z,a)$ is in $C^\infty\left((\bar{\Omega}\times\Omega)-\Delta\right)$

where $\Delta = \{(a,a) : a \in \Omega\}$.

Fact: $K(z,w) = -\frac{2}{\pi} \frac{\partial^2 G}{\partial z \partial \bar{w}}(z,w)$

Note: $0 = \frac{\partial}{\partial \bar{z}} K(z,w) = -\frac{2}{\pi} \underbrace{\frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}}_{\frac{1}{4}\Delta} \frac{\partial}{\partial \bar{w}} G(z,w)$

Pf of Fact: $G(z,w) = -\ln|z-w| + u(z,w)$

$$G_z(z,w) = -\frac{1}{2(z-w)} + u_z(z,w)$$

$$G_{z\bar{w}}(z,w) = 0 + \underbrace{u_{z\bar{w}}(z,w)}_{\text{Analytic in } z,\ \in C^\infty(\bar{\Omega}\times\Omega)}$$

Play: $h(z_0) = \langle h, K_{z_0} \rangle$, $h \in H^2(\Omega)$.

This characterizes K_{z_0}. Now

$$= -\frac{2}{\pi} \iint_{w \in \Omega} G_{z\bar{w}}(z_0, w)\, h(w)\, dA =$$

$$\left[\overline{G_{z\bar{w}}(z,w)} = G_{\bar{z}w}(z,w) \right]$$

$$= -\frac{2}{\pi} \iint_{w \in \Omega} \frac{\partial}{\partial \bar{w}} \Big[G_z(z_0,w) \underset{\substack{\uparrow \\ h \text{ analytic}}}{h}(w) \Big] \Big(\underbrace{\tfrac{1}{2i}\, d\bar{w} \wedge dw}_{dA = dx\,dy} \Big)$$

$$= \lim_{\varepsilon \to 0} \frac{-1}{i\pi} \iint_{\Omega_\varepsilon} \frac{\partial}{\partial \bar{w}} \left[G_z(z_0,w)\, h(w) \right] d\bar{w} \wedge dw$$

$\Omega_\varepsilon = \Omega - D_\varepsilon(z_0)$

$$= \lim_{\varepsilon \to 0} \frac{-1}{i\pi} \int_{b\Omega_\varepsilon} \underbrace{G_z(z_0,w)}_{\substack{=0 \text{ in } w \\ \text{on } b\Omega}} h(w)\, dw$$

$$= \lim_{\varepsilon \to 0} \frac{-1}{i\pi} \left(-\int_{bD_\varepsilon} G_z(z_0,w)\, h(w)\, dw \right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{i\pi} \left(\int_{bD_\varepsilon} \left(\frac{-1}{2(z_0 - w)} + \text{Bdd} \right) h(w)\, dw \right)$$

$$= \frac{2\pi i}{2\pi i} h(z_0) = h(z_0).$$

So this thing is equal to K_{z_0}.

Def^n: $\Lambda(z,w) = -\frac{2}{\pi} \frac{\partial^2 G}{\partial z \partial w}(z,w)$

$$= -\frac{1}{2(z-w)^2} + U(z,w)$$

Properties: $\Lambda(z,w)$ is analytic in z and w, has a double pole in z at $z=w$, $\Lambda(z,w) = \Lambda(w,z)$ (because $G(z,w) = G(w,z)$). $\Lambda \in C^\infty((\bar\Omega \times \Omega) - \Delta)$.

Bergman kernel in area QD: Consider

$G_w(z, w_0)$ for $z \in b\Omega$, $w_0 \in \Omega$.

It is zero in z on $b\Omega$. Let

$z(t)$ parametrize $b\Omega$ in standard sense.

$$G_w(z(t), w_0) \equiv 0$$

Diff'tiate wrt t:

$$\frac{\partial G_w}{\partial z}(z(t), w_0)\, z'(t) + \frac{\partial G_w}{\partial \bar{z}}(z(t), w_0)\, \overline{z'(t)} \equiv 0$$

Divide by $|z'(t)|$ and note $T(z(t)) = \dfrac{z'(t)}{|z'(t)|}$.

$$G_{zw}(z, w_0)\, T(z) = -\, G_{\bar{z}w}(z, w_0)\, \overline{T(z)}$$

$z \in b\Omega$

Conjugate:

$$\underbrace{\overline{G_{zw}(z, w_0)}}_{-\frac{\pi}{2}\Lambda(z, w_0)}\ \overline{T(z)} = -\underbrace{G_{z\bar{w}}}_{-\frac{\pi}{2}K(z,w_0)}(z, w_0)\, T(z)$$

$$\boxed{K(z, w_0)\, T(z) = -\overline{\Lambda(z, w_0)\, T(z)}}$$

$z \in b\Omega$

Consequence: If Ω is an area QD, we know $\sum c_{jm} K_{a_j}^{(m)} \equiv 1$. Use green box:

$$\mathbb{1}\cdot\overline{T(z)} = -\sum c_{jm}\,\overline{\Lambda^m(z,a_j)}\;\overline{T(z)}.$$

Divide green box by this and get

$$K(z,w_0) = \frac{\text{anti-holo}}{\text{anti-holo}}\;\overline{\overline{T}}\big/\overline{\overline{T}}$$

$$= \left(\begin{array}{c}\text{anti-holo}\\ \text{thing}\end{array}\right) \text{ on } b\Omega.$$

Conclude that K_{w_0} extends meromorphically to the double! So K_{w_0} is a rational function of z and $S(z)$! $S(z)$ is algebraic. Hence so is K_{w_0}. $S(z)=\bar{z}$ on $b\Omega$. So K_{w_0} is rational in z and $\bar{z}$ on $b\Omega$.

Fact: This implies that $K(z,w)$ is a rational fcn of $z, S(z), \bar{w}, \overline{S(w)}$.

Similar interesting fact: Suppose $H(z,w)$ is continuous on $D_1(0)\times D_1(0)$, and

Suppose $H(z, w_0)$ is a poly in z for each $w_0 \in D_1(0)$ and $H(z_0, w)$ is a poly in w for each fixed $z_0 \in D_1(0)$. Then

$$H(z,w) = \sum_{n,m \le N} c_{nm} z^n w^m$$ is a poly in z & w.

Similar ideas hold for arc-length QD and the Szegő kernel.

$$(*) \quad S_a = \overline{\tfrac{1}{i} L_a T} = i \, \overline{L_a} \, \overline{T}.$$

bndry arc-length QD $\Rightarrow \sum c_{jm} S_{a_j}^{(m)} \equiv 1$

Get $$1 = i \sum c_{jm} \overline{L_a^{(m)}} \, \overline{T}. \quad (+)$$

Divide $(*)$ by $(+)$ to see

$$S_a = (\text{anti-holo}) \quad \text{on } b\Omega.$$

So S_a extends mero to the double.