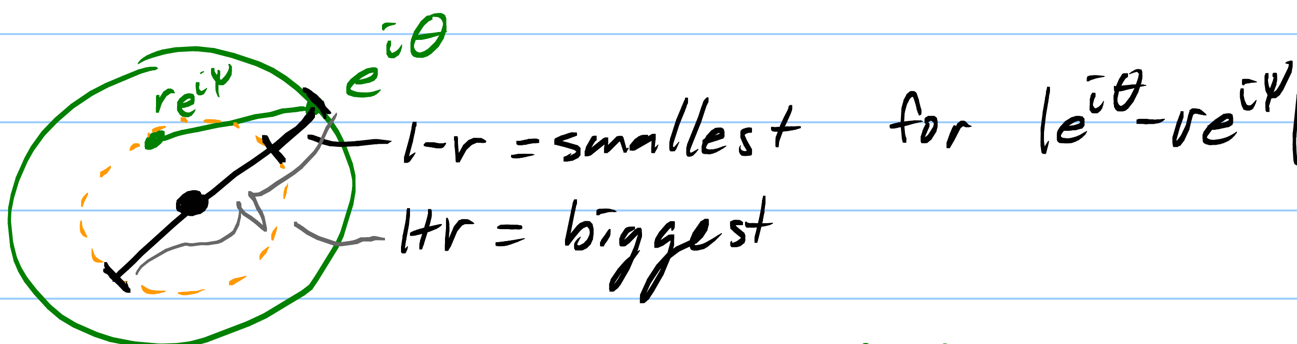
Harnack's Ineq: Suppose u is a positive harmonic fcn on $D_1(0)$ that is continuous on $\overline{D_1(0)}$. Let $z = re^{i\psi}$.

$$\frac{1-r}{1+r}\,u(0) \le u(z) \le \frac{1+r}{1-r}\,u(0)$$

Pf: Poisson kernel $\frac{1}{2\pi}\frac{1-r^2}{|e^{i\theta}-re^{i\psi}|^2} = P(z,\theta)$



Basic estimate $P(z,\theta) = \dfrac{(1-r)(1+r)}{2\pi\,|e^{i\theta}-re^{i\psi}|^2}$

See $\frac{1}{2\pi}\frac{1-r}{1+r} \le P(z,\theta) \le \frac{1}{2\pi}\frac{1+r}{1-r}$

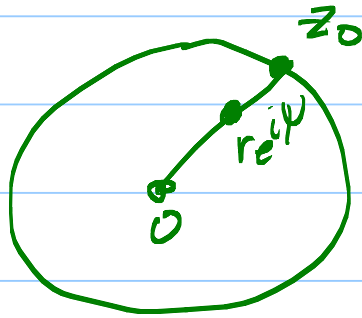
Now integrate against $u(e^{i\theta})\,d\theta$.

Use $\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta = u(0)$. Get

$$\boxed{\frac{1-r}{1+r}\,u(0) \le u(z) \le \frac{1+r}{1-r}\,u(0)} \quad |z|=r.$$

Hopf Lemma: Suppose u is in $C^1(\overline{D_1(0)})$, and u is harmonic on $D_1(0)$. If u attains it's max at z_0, $|z_0| = 1$, and u is not constant, then $\frac{\partial u}{\partial n}(z_0) > 0$.

Pf:



$z = re^{i\psi}$. $M = u(z_0)$.

Suppose u not const. Then $M - u$ is a positive harmonic fcn.

Left side of Harnack's:

$$\frac{1-r}{1+r}\left(M - u(0)\right) \le M - u(z).$$

$$\frac{1}{1+r}\left(M - u(0)\right) \le \frac{M - u(z)}{1-r} \quad \leftarrow \text{DQ for } \frac{\partial}{\partial n}$$

$$\underbrace{\frac{1}{2}\left(M - u(0)\right)}_{>0} \le \frac{1}{1+r}\left(M - u(0)\right) \le \frac{M - u(z)}{1-r}$$

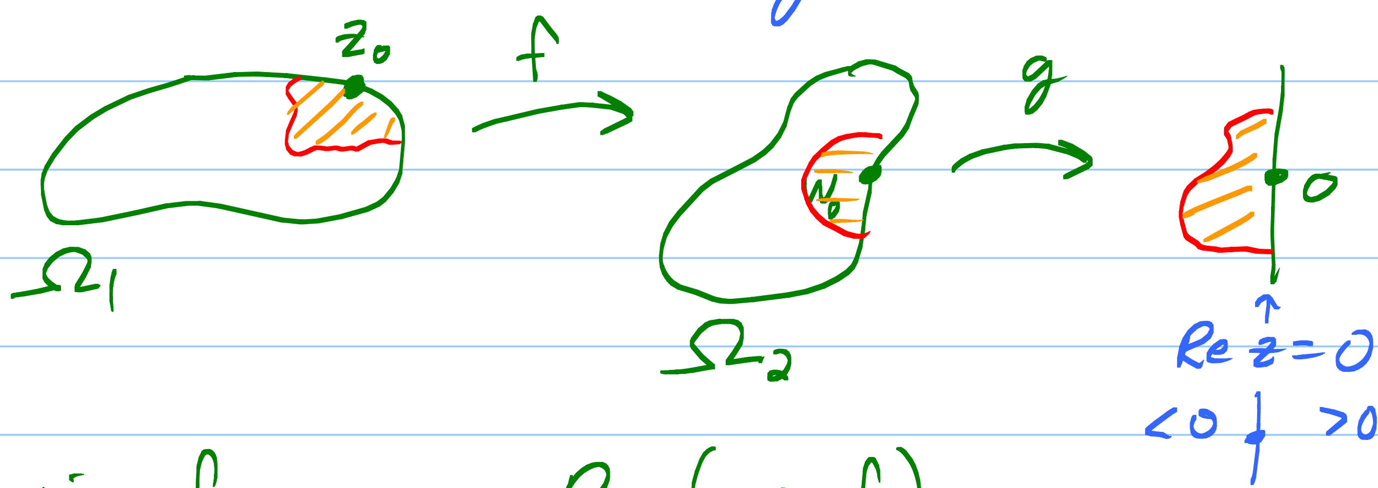
Let $r \nearrow 1$. Conclude $\frac{\partial}{\partial n} > 0$.

Application: Estimate, even without C^1

assumption : $\quad \underbrace{(1-r)}_{\mathrm{dist}(z,b\Omega)} \leq c\,(M-u(z))$

<u>Application</u> : Suppose $f: \Omega_1 \to \Omega_2$ is a conformal map between C^∞ domains. Then we know $f \in C^\infty(\overline{\Omega_1})$. Hopf Lemma $\Rightarrow$ f' is non-vanishing on $b\Omega_1$.

<u>Pf</u>:



Harmonic fcn $\quad u = \mathrm{Re}\,(g \circ f)$.

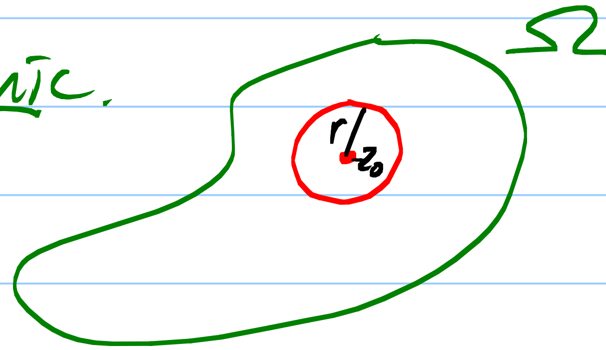
is not const., and attains its max at z_0.

So $\frac{\partial}{\partial n} > 0$. But, if $f'(z_0) = 0$, chain rule yields $\frac{\partial}{\partial n} = 0$. $\lightning$. So $f'(z_0) \neq 0$.

<u>Alternate Pf</u>: Map locally to domains with real analytic bndry. Schwarz reflection

shows maps in new coord extend analytically past boundary. If $f'(z_0) = 0$, map looks like $z^n: 0 \mapsto 0$. n must $= 1$ to map one side to one side.

Remark: Important left side of Harnack's works for u subharmonic.



$$u(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

Fact: u C^2 smooth is subharmonic $\Leftrightarrow \Delta u \ge 0$.

How to cook up a subharmonic defining fcn for smooth Ω: Solve $\begin{cases} \Delta\rho \equiv 1 \\ \rho|_{b\Omega} = 0 \end{cases}$

Harnack's ineq: $\rho(z) \le -c \operatorname{dist}(z, b\Omega)$.

$f: \underset{\rho_1}{\overset{\uparrow}{\Omega_1}} \to \underset{\rho_2}{\overset{\uparrow}{\Omega_2}}$

Use Harnack's:

$\operatorname{dist}(f(z), b\Omega_2) \le c \operatorname{dist}(z, b\Omega_1)$.

Outline of pf that biholo between strictly pseudoconvex with smooth bndry extend smoothly to bndry.

<u>Step 1</u>: $f : \Omega_1 \to \Omega_2$ domains in $\mathbb{C}^n$.

$$u = \det\left[\frac{\partial f_i}{\partial z_j}\right] \underset{\text{linear alg.}}{=} \left| \begin{matrix}\text{Jacobian of } f : \mathbb{R}^{2n} \\ \text{det} \qquad\qquad \to \mathbb{R}^{2n}\end{matrix} \right|^2$$

Conclude $\varphi \mapsto u(\varphi \circ f)$ is an isometry

$$L^2(\Omega_2) \to L^2(\Omega_1).$$

$$\boxed{P_1\big(u(\varphi \circ f)\big) = u \cdot \big((P_2\varphi) \circ f\big)}$$

P_1, P_2 are Bergman projections.

<u>Step 2</u>: Strictly pseudoconvex domains have strictly plurisubharmonic defining functions.

<u>Step 3</u>: Harnack's Ineq argument works

in this setting. Get

$$c_1 \operatorname{dist}(z, b\Omega_1) \leq \operatorname{dist}(f(z), b\Omega_2) \leq c_2 \operatorname{dist}(z, b\Omega_1).$$