Step 4: $f : \Omega_1 \to \Omega_2$ biholo between C^∞ smooth strictly pseudoconvex domains in $\mathbb{C}^n$. (C. Fefferman)

Bergman Projection $P_j : C^\infty(\overline{\Omega_j}) \circlearrowleft$

In $\mathbb{C}^1$: $P = I - 4 \frac{\partial}{\partial z} G \frac{\partial}{\partial \bar z}$, "Elliptic PDE."

In $\mathbb{C}^n$: $P = I - \vartheta N \bar\partial$, "Subelliptic PDE problem"

↑

$\bar\partial$-Neumann operator

$\bar\partial u = \sum \frac{\partial u}{\partial \bar z_j} d\bar z_j$ ← (0,1)-form

↖ $\bar\partial$-operator.

L^2-theory: $\langle \bar\partial u, \alpha \rangle = \langle u, \vartheta \alpha \rangle_{L^2(\Omega)}$

↑ ↗ (0,1)-forms ↑ ↗ functions

If $\alpha = \sum \alpha_j d\bar z_j$, then $\vartheta \alpha = -\sum \frac{\partial \alpha_j}{\partial \bar z_j}$.

Kohn 60s showed regularity up to $b\Omega$ for the $\bar\partial$-Neumann problem.

$P : C^\infty(\bar\Omega) \circlearrowleft$

Catlin: extended Kohn's theorem to

weakly pseudoconvex domains of "finite type."

Step 5: If $\varphi \in C^\infty(\bar{\Omega})$ and $\varphi|_{b\Omega} \equiv 0$,

then $\frac{\partial \varphi}{\partial z_j} \perp H^2(\Omega)$.

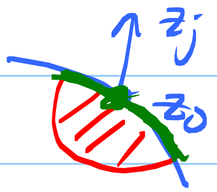
$$\left\langle h, \frac{\partial \varphi}{\partial z_j} \right\rangle = \int_\Omega h \overline{\frac{\partial \varphi}{\partial z_j}} \, dV$$

$$= \int_\Omega h \frac{\partial \bar{\varphi}}{\partial \bar{z}_j} \, dV = - \int_\Omega \underbrace{\frac{\partial h}{\partial \bar{z}_j}}_{0} \bar{\varphi} \, dV = 0.$$

Let ρ be a plurisubharmonic defining function for Ω. Then $\nabla \rho \neq 0$ on $b\Omega$.

So $\exists$ complex direction z_j so that, given $z_0 \in \Omega$, $\frac{\partial \rho}{\partial z_j}(z_0) \neq 0$.

Lemma: Given $v \in C^\infty(\bar{\Omega})$, can cook up $\varphi \in C^\infty(\bar{\Omega})$ with $\varphi|_{b\Omega} = 0$ near z_0 and $v - \frac{\partial \varphi}{\partial z_j}$ vanishing to order N on $b\Omega$ near z_0.



$\frac{\partial \rho}{\partial z_j} \neq 0$.

Next, use a partition of unity:

get $\Psi = \sum_{k,j} \frac{\partial(\chi_k \varphi_k)}{\partial z_j} \perp H^2(\Omega)$

with $v - \Psi$ vanishing to high order on $b\Omega$.

Step 6: Take $h \equiv 1$. Get $\Psi \perp H^2(\Omega_2)$ which agrees with h to high order on $b\Omega_2$.

$$u = \det\left[\frac{\partial f_i}{\partial z_j}\right].$$

$$u = u \cdot (h \circ f) = u \cdot \left[\underbrace{(P_2 h)}_{Ph = h} \circ f\right] =$$

$$u \cdot \left[P_2(\underbrace{h - \Psi}_{\substack{\varphi \\ \text{vanishes to} \\ \text{order } N \text{ on } b\Omega}}) \circ f\right] = P_1(u(\varphi \circ f))$$

Big Idea: φ vanishes to high order. Hence maybe so does $u(\varphi \circ f)$. So $u(\varphi \circ f)$ has some smoothness up to $b\Omega_1$. Kohn $\Rightarrow$ $P_1(u(\varphi \circ f))$ smooth up to $b\Omega_1$.

Conclude $u \in C^\infty(\overline{\Omega}_1)$.

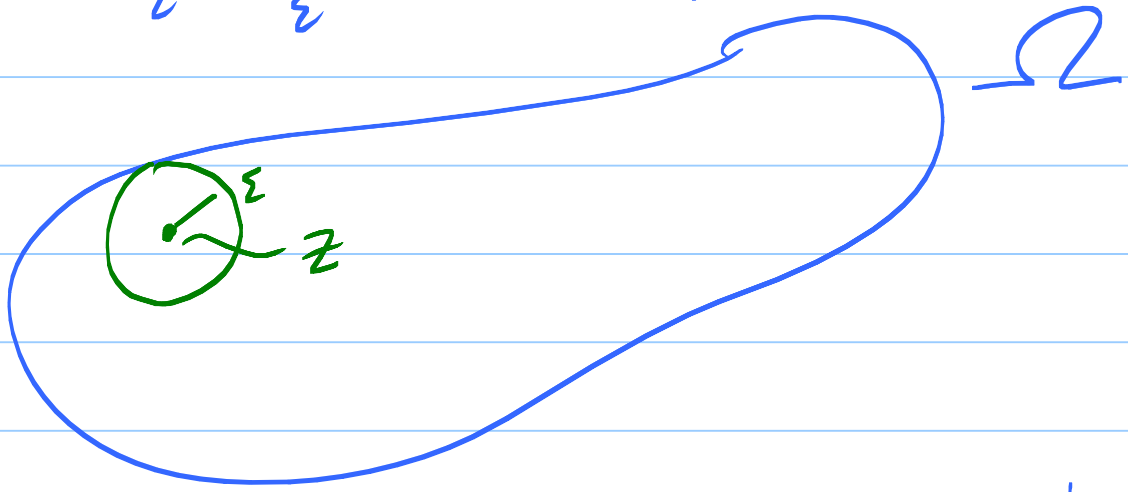
Step 7: **Fact:** f BOUNDED. Then

$$\frac{\partial f}{\partial z_j}(z) \le C\,\mathrm{dist}(z, b\Omega)^{-1}.$$

Why: $\theta \in C_0^\infty(B_1(0))$ radially sym., ≥ 0, and $\int_{\mathbb{C}^n} \theta \, dV = 1$.

Easy fact: $\int_{\mathbb{C}^n} h\theta \, dV = h(0)$ if h holo on $B_1(0)$.

Let $\theta_\varepsilon(z) = \frac{1}{\varepsilon^{2n}}\theta(z/\varepsilon)$



$$\left|\frac{\partial h}{\partial z_j}(z_0)\right| = \left|\int_\Omega \frac{\partial h}{\partial z_j}\,\theta_\varepsilon(z-z_0)\,dV\right| =$$

$$= \left| - \int_\Omega h \underbrace{\frac{\partial}{\partial z_j} \theta_\varepsilon(\cdot - z_0)}_{\frac{1}{\varepsilon} \cdot (\text{something in } L^1)} \, dV \right|$$

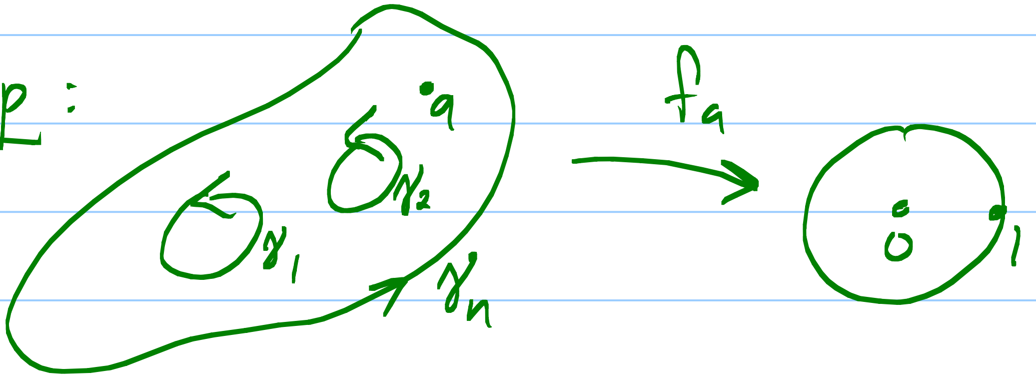
Conclude: $\left| \frac{\partial h}{\partial z_j}(z_0) \right| \le c \, \text{dist}(z_0, b\Omega)^{-1}$!

Consequently: $|u| = \left| \det \left[\frac{\partial f_i}{\partial z_j} \right] \right| \le c \, \text{dist}(z, b\Omega)^{-n}$

Step 8: $u(\varphi \circ f)$ $\qquad \varphi \le c \, \text{dist}(z, b\Omega_2)^N$

Use chain rule for derivatives of $u(\varphi \circ f)$. Can choose N big enough to wipe out finite order blowing up that develops. Also use estimate $\text{dist}(f(z), b\Omega_2) \le c \, \text{dist}(z, b\Omega_1)$ that comes from Harnack's.

Ahlfors Map:



Ω n-connected C^∞ smooth

$f_a : \Omega \to D_1(0)$ is n-to-1 counted with multiplicity and <u>onto</u>, $f_a \in C^\infty(\bar{\Omega})$,

$$f_a : \gamma_j \xrightarrow[\text{onto}]{1-1} \{|z|=1\}.$$

f_a solves the extremal problem:

$$\tilde{\mathcal{F}} = \{h : \Omega \to D_1(0), \text{ analytic}\}.$$

$$\mathcal{M} = \sup\{|h'(a)| : h \in \tilde{\mathcal{F}}\}.$$

$$f_a'(a) = \mathcal{M}, \quad f_a(0) = 0.$$