Step 7: $f : \Omega_1 \to \Omega_2$

$$u = \det\left[\partial f_i / \partial z_j \right]$$

$$u = u(1 \circ f) = u\Big[(\underbrace{P_2 1}_{=\, P_2(\underbrace{1-\psi}_{\varphi})}) \circ f \Big]$$

φ vanishes to high order on $b\Omega$. $\psi \perp H^2$

$$= u\big[(P_2 \varphi) \circ f \big] = P_1\big(\underbrace{u(\varphi \circ f)}_{C^s(\overline{\Omega}_1)} \big) \in C^\infty(\overline{\Omega}_1)$$

s as big as desired

Not done yet!

Step 8: Let $F = f^{-1}$. Let

$$U = \det\left[\frac{\partial F_i}{\partial w_j} \right].$$

Repeat above. Get $U \in C^\infty(\overline{\Omega}_2)$.

Hmmm. $u(z) = \dfrac{1}{U(f(z))}$

Claim: This shows that $u(z_0) \neq 0$ when $z_0 \in b\Omega_1$.

Pf: Take $z_K \in \Omega_1$ with $z_K \to z_0 \in b\Omega_1$.

Ω_2 bounded. So $\{f(z_K)\}$ is bounded.

Can take subseq so that $z_K \to z_0$ and

$f(z_K) \to w_0$. Clear that $w_0 \in b\Omega_2$.

[If $w_0 \in \Omega_2$, then $F(w_0) = z_0$. But $F(B_\epsilon(w_0))$ is a subdomain of Ω_1. So $z_0 \in \Omega_1$. $\lightning$]

$$u(z_K) = \frac{1}{U(f(z_K))}$$

$$\downarrow \qquad \qquad \downarrow$$

$$u(z_0) = 1/U(w_0)$$

$\leftarrow$ $U(w_0)$ can't be 0 because u bdd on $\overline{\Omega_1}$.

So $u(z_0) \neq 0$ too.

Conclude: u non-vanishing on $\overline{\Omega_1}$.

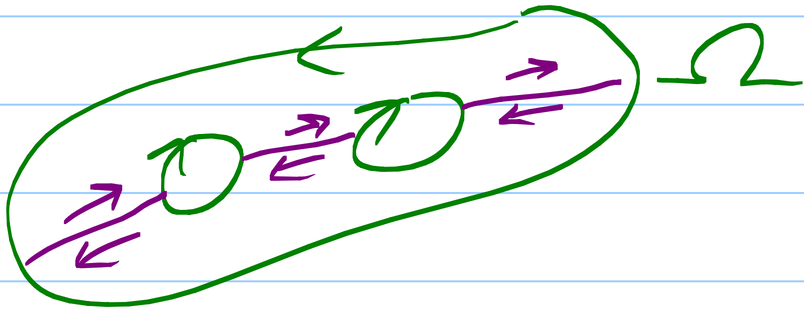
<u>Step 9</u>: $u f_i = u(z_i \circ f)$

$$= u[(P_2 z_i) \circ f] = u[P_2(\underbrace{z_i - \Psi}_{\varphi}) \circ f]$$

$$= P_1(u(\varphi \circ f)) \in C^\infty(\overline{\Omega_1}).$$

<u>Step 10</u>: $f_i = \frac{u f_i}{u}$ ← non-vanish ✓

Extension of the Argument Principle:



f mero on on nbhd of $\bar{\Omega}$, no poles or zeroes on $b\Omega$.

$$\frac{1}{2\pi}\Delta_{b\Omega}\arg f = \begin{pmatrix}\#\text{ zeroes of}\\ f \text{ in } \Omega\end{pmatrix} - \begin{pmatrix}\#\text{ poles of}\\ f \text{ in } \Omega\end{pmatrix}$$

<u>Fancier Version</u>: Assume $b\Omega$ real analytic.

$$\frac{1}{2\pi}\Delta_{b\Omega}\arg f = \left(\text{As above}\right) + \frac{1}{2}\begin{pmatrix}\#\text{ zeroes of } f\\ \text{on } b\Omega\end{pmatrix} - \frac{1}{2}\begin{pmatrix}\#\text{ poles of } f\\ \text{on } b\Omega\end{pmatrix}$$