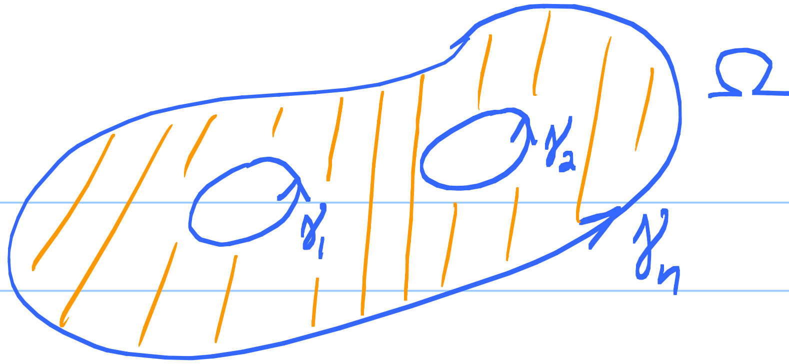
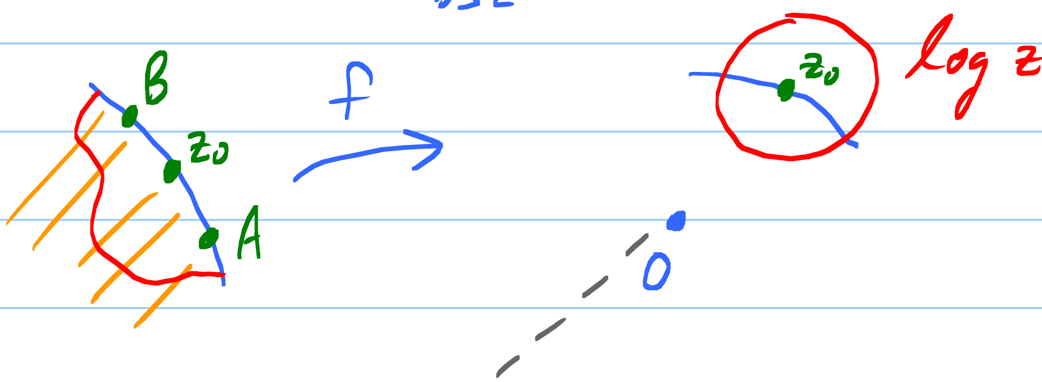


f analytic on $\overline{\Omega}$, no zeroes or poles on $b\Omega$.

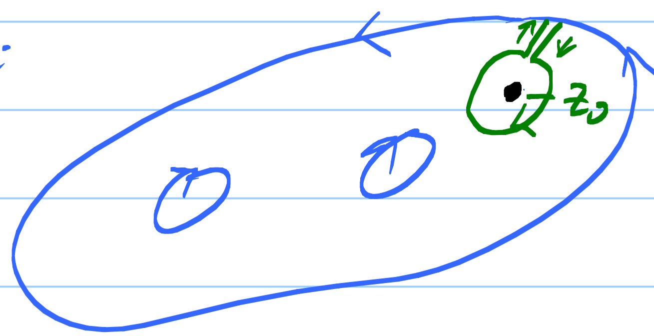
$$\frac{1}{2\pi}\Delta_{b\Omega}\arg f = \frac{1}{2\pi i}\int_{b\Omega}\frac{f'}{f}\,dz = \begin{pmatrix}\#\text{ zeroes}\\ f\\ \text{in } \Omega\end{pmatrix} - \begin{pmatrix}\#\text{ poles}\\ f\\ \text{in } \Omega\end{pmatrix}$$



$$\int_{\gamma_A^B}\frac{f'}{f}\,dz = \int_{\gamma_A^B}\frac{d}{dz}\left(\log f\right)dz = \log f(B) - \log f(A)$$

$$\log f(z) = \ln|f(z)| + i\arg f(z)$$

Stein's book:



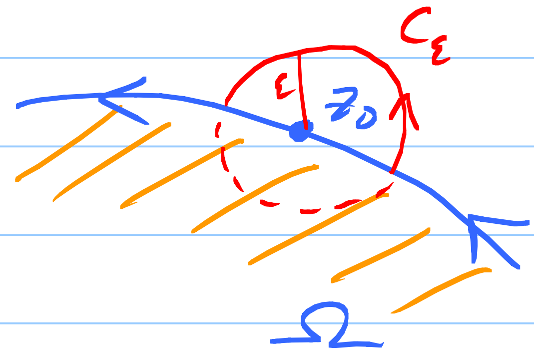
z_0 zero or pole

$$\frac{f'}{f} = \frac{m}{z-z_0} + (\text{analytic})$$

z_0 zero: $m = \text{mult.}$

z_0 pole: $m = -\text{order}$

<u>Fancier version</u>: Assume basic version proof above.



$$\Omega_\varepsilon = \Omega \cup D_\varepsilon(z_0)$$

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{f'}{f}\,dz = \lim_{\varepsilon\to 0} \int_{\alpha}^{\alpha+\pi} \left(\frac{m}{(z_0+\varepsilon e^{i\theta})-z_0} + \text{Bdd} \right) i\varepsilon e^{i\theta}\,d\theta$$

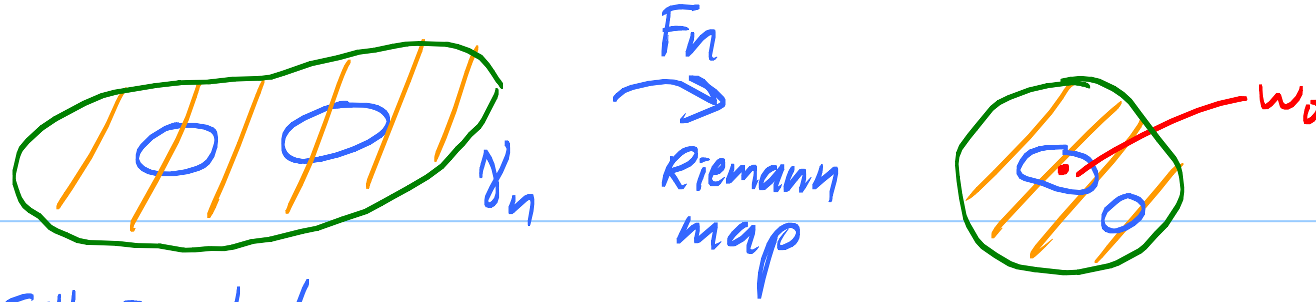
$$= \pi i m$$

Conclude: zeroes and poles on $b\Omega$ contribute <u>half</u> the usual. (Note: $\frac{1}{2\pi i}(\pi i m) = \frac{1}{2} m$)

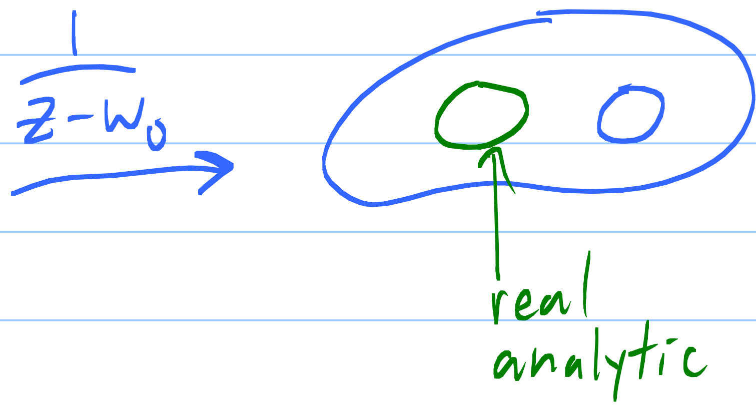
<u>Note</u>: $\frac{1}{2\pi} \Delta_{b\Omega} \arg f = \lim_{\varepsilon_j \to 0} \frac{1}{2\pi} \Delta_\varepsilon \arg f$

where $\Delta_\varepsilon \arg f$ = increase in $\arg f$ as we go around $b\Omega$ <u>minus</u> punched out discs $D_{\varepsilon_j}(z_j)$.

<u>Ahlfors Map</u>: <u>Step 1</u>: Map a smooth n-connected domain to one with real analytic boundary.



Fill in holes.



Repeat!

Step 2: May assume Ω real analytic boundary. Know S_a and L_a extend holo past $b\Omega$.

Plan: Show that $f_a = \frac{S_a}{L_a}$ is the "Ahlfors Map." Only tool: $\boxed{\overline{S_a} = \frac{1}{i} L_a T}$ (*)

Step 3: Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Want to show $\exists$ exactly one point on each γ_j such that f_a maps to λ.

Think: $f_a(z_0) = \dfrac{S_a(z_0)}{L_a(z_0)} = \lambda$

Define: $G = S_a - \lambda L_a$

Step 4: Study zeroes of G.

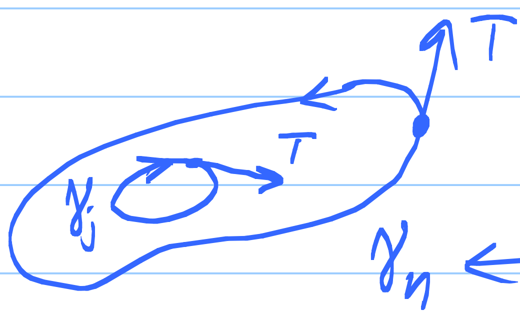
Note:
$$GT = S_a T - \lambda L_a T \overset{(*)}{=}$$
$$= i\bar{L}_a - \lambda i \bar{S}_a$$
$$= i \underbrace{\lambda\bar{\lambda}}_{1} \bar{L}_a - \lambda i \bar{S}_a$$
$$= -\lambda i \left(\bar{S}_a - \bar{\lambda} \bar{L}_a \right)$$

Hmmm. $\quad GT = -\lambda i \, \bar{G} \qquad T = 1/\bar{T}$

$$G = -\lambda i \, \bar{G}\,\bar{T}$$

Multiply by G: $\quad \boxed{G^2 = -\lambda i \, |G|^2 \, \bar{T}} \quad (\dagger)$

Facts:



$\gamma_n \leftarrow \Delta_{\gamma_n} \arg T = +2\pi$

$$\Delta_{\gamma_j} \arg T = -2\pi$$

$$\Delta \arg \bar{T} = -\Delta \arg T$$

$$\frac{1}{2\pi} \Delta_{b\Omega} \arg T = (+1) + (n-1)(-1) = 2-n$$

$$\frac{1}{2\pi}\Delta_{b\Omega}\arg \overline{T} = n-2.$$

First consequence of $(+)$: G must have at least one zero on each boundary curve.

If not $\quad G^2 = -\lambda i \underbrace{|G|^2\overline{T}}$

Δ arg of a square around any γ_K is <u>even</u> multiple of 2π.

Δ arg is $\pm 2\pi$

No way!

<u>Next</u>: Apply beefed up arg princ to G.

$$G = S_a - \lambda L_a$$

simple pole at a.

at least one zero on each bndry curve

$$\frac{1}{2\pi}\Delta_{b\Omega}\arg G = \underset{\text{pole}}{(-1)} + \underbrace{\begin{pmatrix}\#\text{ zeroes } G\\ \text{in } \Omega\end{pmatrix}}_{\text{must }=0} + \frac{1}{2}\begin{pmatrix} n + \text{extra}\\ \text{zeroes}\\ \text{on } b\Omega\end{pmatrix}$$

extra stuff not there

$$= \frac{1}{2\pi}\frac{1}{2}\Delta_{b\Omega}\arg\overline{T} = \frac{n-2}{2}$$

$G^2 = -\lambda i |G|^2 \overline{T}$

Conclude: G has exactly one simple zeroe on each boundary curve and no zeroes inside Ω.

Step: Now look at $f_a = \dfrac{S_a}{L_a}$.

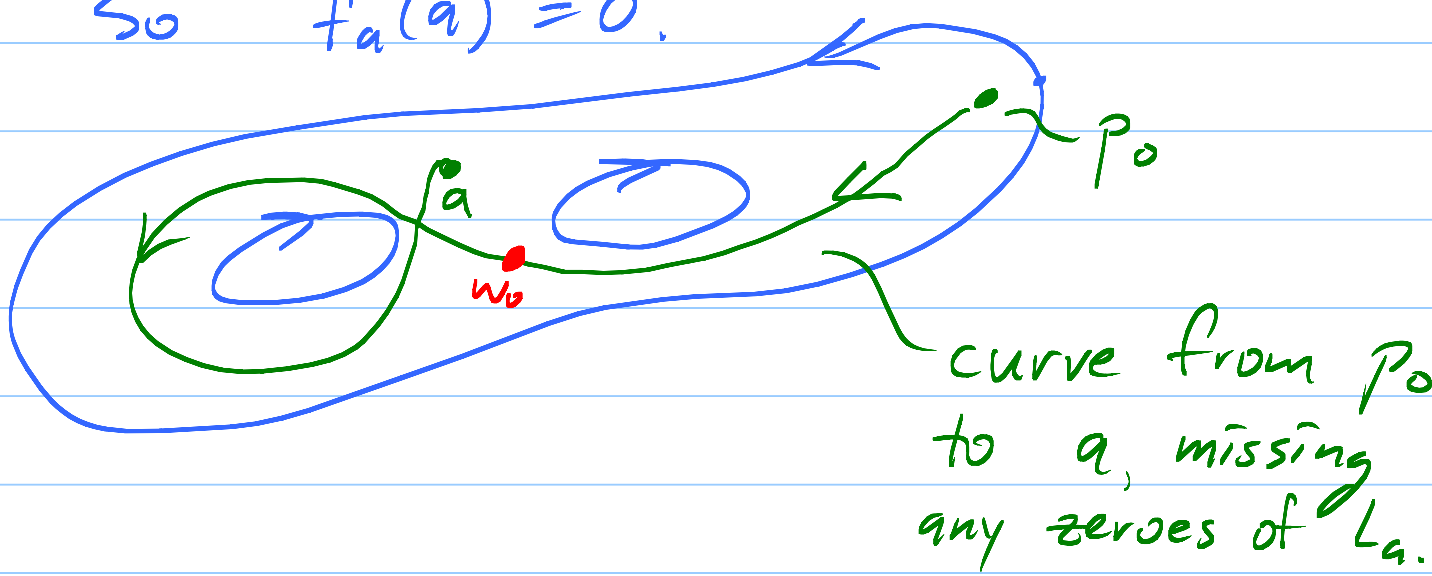
First: No such $p_0 \in \bar{\Omega}$ where $|f_a(p_0)| > 1$.

Note: If $p_0 \in b\Omega$, slide into Ω to get a $p_0 \in \Omega$ where $|f_a(p_0)| > 1$. May also assume p_0 is not a zero of L_a.

We know 1) $S_a(a) = S(a,a) = \|S_a\|^2_{L^2(b\Omega)} > 0$.

2) L_a has a simple pole at a.

So $f_a(a) = 0$.



Intermediate value thm yields w_0 on curve where $f_a(w_0) = \lambda$ where $|\lambda| = 1$.

Oops. With this λ, $G(w_0) = 0$. $\lightning$.