Ahlfors map. Fix $a \in \Omega$. $f_a = \frac{S_a}{L_a}$

Last time: showed no $p_0 \in \overline{\Omega}$ with $|f(p_0)| > 1$.

<u>Conclude</u>: f_a has no poles. [If it did, easy to get p_0 nearby where $|f_a(p_0)| > 1$. $\lightning$.]

We now know: $|f_a(z)| \leq 1$ on $\overline{\Omega}$ and

$|f_a(z)| = 1$ for $z \in b\Omega$. [$\overline{S_a} = \frac{1}{i} L_a T$. So

$\left|\frac{S_a}{L_a}\right| = |T| = 1$ on $b\Omega$.]

We also know $f_a(a) = 0$. Max Princ. $\Longrightarrow$

Hence $|f_a(z)| < 1$ for $z \in \Omega$ and $= 1$ for $z \in b\Omega$.

<u>Step</u>: $L_a(z) \neq 0$ for $z \in \Omega - \{a\}$.

<u>Pf</u>: If $L_a(z_0) = 0$ for $z_0 \in \Omega - \{a\}$, then

$S_a(z_0) = 0$ too (since $|f_a(z_0)| < 1$).

Oops! Now $G_\lambda(z_0) = S_a(z_0) - \lambda L_a(z_0) = 0$

too. But G_λ has no zeroes in Ω. $\lightning$.

<u>Step</u>: $L_a(z) \neq 0$ for $z \in b\Omega$.

<u>Pf</u>: As above, we see $S_a(z_0) = 0$ too at

a zero z_0 of L_a in $b\Omega$. Conclude

$G_\lambda(z_0) = 0$ too for <u>any</u> λ with $|\lambda| = 1$.

Suppose $z_0 \in \gamma_j$. We know $S_a \not\equiv 0$.

So $\exists$ $w_0 \in \gamma_j$ where $S_a(w_0) \neq 0$.

$L_a(w_0) \neq 0$ too $\left[\bar{S}_a = \frac{1}{i} L_a T \right]$.

Let $\lambda_0 = \dfrac{S_a(w_0)}{L_a(w_0)}$. Oops! G_{λ_0}

vanishes at <u>two</u> points z_0 and w_0 in γ_j. $\lightning$.

We have shown $L_a(z)$ has <u>no</u> zeroes in $\bar{\Omega} - \{a\}$.

Next: Show that S_a has exactly $n-1$ zeroes (counted with mult.) in Ω and no zeroes on $b\Omega$.

<u>Pf</u>: No zeroes on $b\Omega$ clear from $\bar{S}_a = \frac{1}{i} L_a T$ and no zeroes of L_a on $b\Omega$. Now use Argument Principle and $\bar{S}_a = \frac{1}{i} L_a T$

$$\frac{1}{2\pi} \Delta_{b\Omega} \arg \bar{S}_a = \frac{1}{2\pi} \Delta_{b\Omega} \arg L_a + \frac{1}{2\pi} \Delta_{b\Omega} \arg T$$

$$-(\#\text{zeroes of } S_a \text{ in } \Omega) = (-1 + 0) + (2 - n)$$

$\left(\#\text{zeroes } S_a \text{ in } \Omega\right) = n-1.$

We can now say: $f_a = \frac{S_a}{L_a}$ has n zeroes in Ω. ($n-1$ at zeroes of S_a and one at pole a of L_a.)

$S_a(a) = S(a,a) = \|S_a\|^2_{L^2(b\Omega)} \neq 0$

Step: $f_a : \Omega \to D_1(0)$ is n-to-1 and onto. (Counted with mult.)

Pf: $N(w) = \left(\#\text{times } f_a(z) = w \text{ in } \Omega\right)$, $w \in D_1(0)$.

We know $N(0) = n$.

Arg. Princ. $N(w) = \frac{1}{2\pi i} \int_{b\Omega} \frac{f_a'(z)}{f_a(z) - w}\, dz$

is analytic in w and integer valued, so constant. So $N(w) \equiv n$.

Last thing: Show f_a solves extremal prob: $f_a'(a)$ is real and maximum $|h'(a)|$ among $\tilde{\mathcal{F}} = \{h : \Omega \to D_1(0), \text{ analytic}\}$.

Pf: $\overline{S_a} = \frac{1}{i} L_a T$ implies $\boxed{\overline{S_a}^2\, \overline{T} = -L_a^2\, T}$

Claim: $\mathrm{Res}_a\, L_a^2 = 0.$

because $\mathrm{Res}_a\, L_a^2 = \frac{1}{2\pi i}\int_{b\Omega} L_a^2\, dz$

$$= \frac{1}{2\pi i}\int_{b\Omega} L_a^2\, T\, ds = -\frac{1}{2\pi i}\int_{b\Omega} \bar{S}_a^2\, \bar{T}\, ds$$

$$= -\frac{1}{2\pi i}\overline{\int_{b\Omega} S_a^2\, \underbrace{T\, ds}_{dz}} = 0 \text{ by Cauchy's Thm.}$$

Conclude $\quad L_a(z)^2 = \frac{1}{4\pi^2 (z-a)^2} + \underset{\substack{\uparrow \\ \text{analytic}}}{H_a}$

Now suppose $h : \Omega \to D_1(0)$.

Fact: h has L^2 boundary values and basic formulas all hold.

$$h'(a) = \mathrm{Res}_a \frac{h}{(z-a)^2} = \frac{1}{2\pi i}\int_{b\Omega} \frac{h}{(z-a)^2}\, dz$$

$$= 4\pi^2 \cdot \frac{1}{2\pi i}\int_{b\Omega} h \left[\frac{1}{4\pi^2 (z-a)^2} + \underset{\substack{\uparrow \\ \text{adds } 0}}{H_a}\right] dz$$

So $h'(a) = \frac{2\pi}{i} \int_{b\Omega} h \underbrace{L_a^2 \, dz}_{L_a^2 T \, ds}$

Now $\left| h'(a) \right| \le 2\pi \int_{b\Omega} \underbrace{|h|}_{\le 1 \text{ on } b\Omega} \left| L_a^2 \right| ds$

$\le 2\pi \int_{b\Omega} \underbrace{|L_a|^2}_{= |S_a|^2} ds = 2\pi \langle S_a, S_a \rangle$

$= 2\pi S(a,a) \longleftarrow = f_a'(a)$!

<u>Note</u>: $f_a = \frac{S_a}{L_a}$ and $L_a(z) = \frac{1}{2\pi(z-a)}$ + analytic.

So $f_a'(a) = \frac{S_a(a)}{(1/2\pi)} = 2\pi S(a,a)$. ✓

Conclude f_a is ~~the~~ extremal.

<u>Claim</u>: Extremal is unique. (Next time.)

<u>Note</u>: Proved along the way. G_a has exactly one simple zero on each boundary curve ($|h|=1$). This shows that $f_a : \gamma_j \xrightarrow[\text{onto}]{1-1} \{|z|=1\}$.