To show f_a is <u>unique</u> sol^n to extremal prob:

$h : \Omega \to D_1(0)$. Suppose h extremal, $h'(a) \in \mathbb{R}^+$. [replace h by $e^{i\theta} h$ for this.] Know

$$h'(a) = \frac{2\pi}{i} \int_{b\Omega} L_a^2 \, h \, T \, ds \le 2\pi \int_{b\Omega} |L_a|^2 ds = 2\pi S(a,a)$$

Note: Since $h : \Omega \to D_1(0)$, $|h| \le 1$ on $b\Omega$ a.e.

Measure Theory Fact: If $|u| \le |v|$ a.e. and

$$\int_{b\Omega} u \, ds = \int_{b\Omega} |v| \, ds, \quad \text{then } u = |v|.$$

Here $u = \frac{2\pi}{i} L_a^2 \, h \, T$, $\quad v = 2\pi |L_a|^2$.

Then $|u| = 2\pi |L_a|^2 |h| \le 2\pi |L_a|^2 = |v|$.

Conclude $\quad \frac{2\pi}{i} L_a^2 \, h \, T = 2\pi |L_a|^2$ on $b\Omega$.

$$h = i \frac{L_a \overline{L_a} \, \overline{T}}{L_a^2} = \frac{i \overline{L_a} \, \overline{T}}{L_a} = f_a \checkmark$$

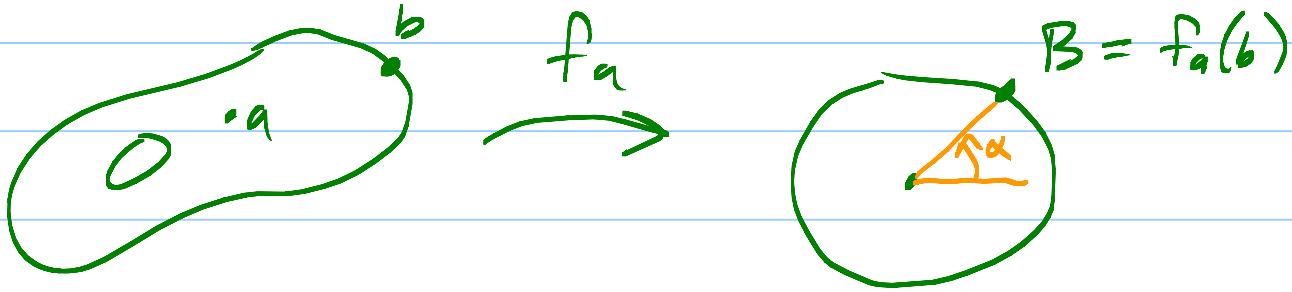
$i \overline{L_a} \, \overline{T} = S_a$

Other facts that follow

1) $f_a \in C^\infty(\overline{\Omega})$. In fact f_a extends

analytically to an open set containing $\bar{\Omega}$.

2) f_a' does not vanish at boundary points.



Harmonic function $u = \operatorname{Re} e^{-i\alpha} f_a$

on Ω, in $C^\infty(\bar{\Omega})$, and it assumes max of 1 at b. Hopf Lemma implies $\frac{\partial}{\partial n} u > 0$ at b. Chain Rule yields that $f_a'(b) \neq 0$.

<u>Zeroes of f_a'</u>: Parametrize $b\Omega$: $z(t)$

$f(z(t))$ parametrizes $\{|z|=1\}$.

$$\frac{f'(z(t))\, z'(t)}{|f'(z(t))|\, |z'(t)|} = \frac{\frac{d}{dt} f(z(t))}{\left|\frac{d}{dt} f(z(t))\right|}$$

$$\frac{f'(z)}{|f'(z)|}\, T_\Omega(z) = T_{D_1}(f(z))$$

Add up Δ arg arround each boundary curve and use conformality on boundaries:

$$\frac{1}{2\pi}\left[\Delta \arg f' + \Delta \arg T_\Omega = \Delta \arg T_{D_1} \circ f\right]$$

Use Arg Princ. Get

Fact: f_a' has $2n-2$ zeroes in Ω counted with multiplicity.

In $\mathbb{C}^n$: $f:(z,w) \mapsto (z,w^2)$ maps

$\{(z,w) : |z|^2+|w|^4<1\} \longrightarrow$ Unit Ball

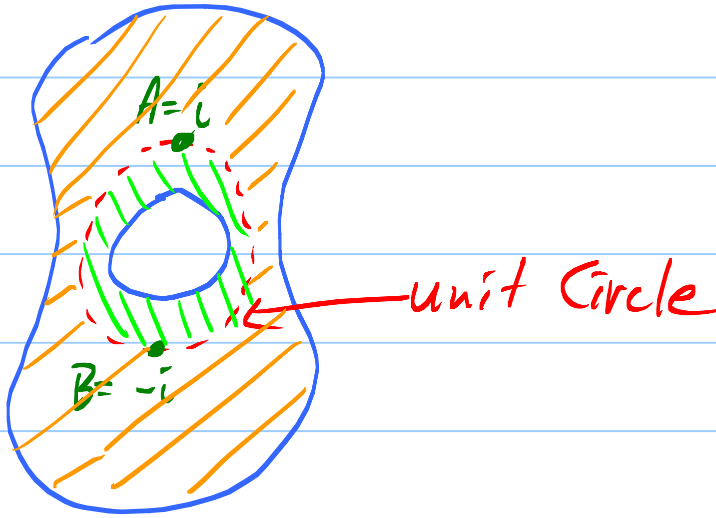
like an Ahlfors Map. $u=\det[f']$ vanishes along $(z,0)$. Zeroes propagate out to $b\Omega$ by Hartog's Thm.

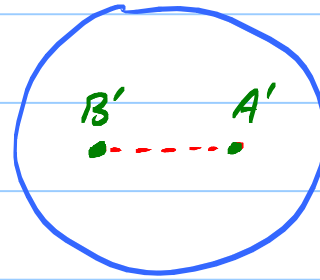
Ex: $\Omega=\{z\in\mathbb{C} : |z+\frac{1}{z}|<r\}$, $r>2$.

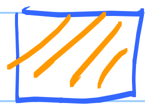
$f(z)=\frac{1}{r}(z+\frac{1}{z})$. $f:\Omega \longrightarrow D_1(0)$

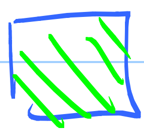
and $f: b\Omega \to \{|z|=1\}$.

Fact: (Ersin Deger) f = Ahlfors map for $a=i$.





f:  $\xrightarrow[\text{onto}]{1-1}$ $D_1(0) - [B', A']$

f:  $\longrightarrow$ same.

$f'(\pm i) = 0$ (simple zeroes)