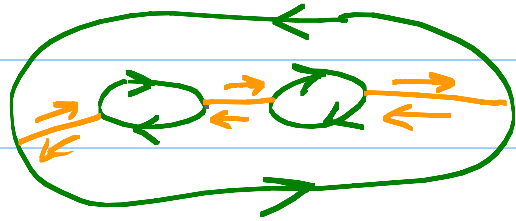
Ahlfors map: $\frac{f'(z)}{|f'(z)|} T_\Omega(z) = T_{D_1}(f(z))$ (*)



(*) shows that

$$\underbrace{\frac{1}{2\pi} \Delta_{b\Omega} \arg f'}_{\substack{\#\text{ zeroes of} \\ f' \text{ in } \Omega}} + \underbrace{\frac{1}{2\pi} \Delta_{b\Omega} \arg T_\Omega}_{1-(n-1)} = \underbrace{\frac{1}{2\pi} \sum \Delta \arg T_D(f)}_{n}$$

Get $\left(\begin{matrix}\#\text{ zeroes of } f_a' \\ \text{in } \Omega\end{matrix}\right) = 2n-2.$

Def: $f: \Omega_1 \to \Omega_2$ is called proper

if $f^{-1}(K)$ is compact in Ω_1 whenever

K is compact in Ω_2.

Fact: If Ω_1 and Ω_2 are smoothly bounded, and f is continuous up to $\overline{\Omega_1}$, analytic on Ω_1, then f is proper $\iff f: b\Omega_1 \to b\Omega_2$.

Fact: If $f: \Omega_1 \to \Omega_2$ is a proper holo map between bounded domains in $\mathbb{C}$, then there is a positive integer m such that f is an m-to-one map (counting multiplicity).

Theorem: Suppose $f: \Omega_1 \to \Omega_2$ is a proper holo map between bounded C^∞ smooth domains in $\mathbb{C}$. The Bergman projections transform

$$P_1\left(f' \cdot (u \circ f)\right) = f' \cdot \left[(P_2 u) \circ f\right]$$

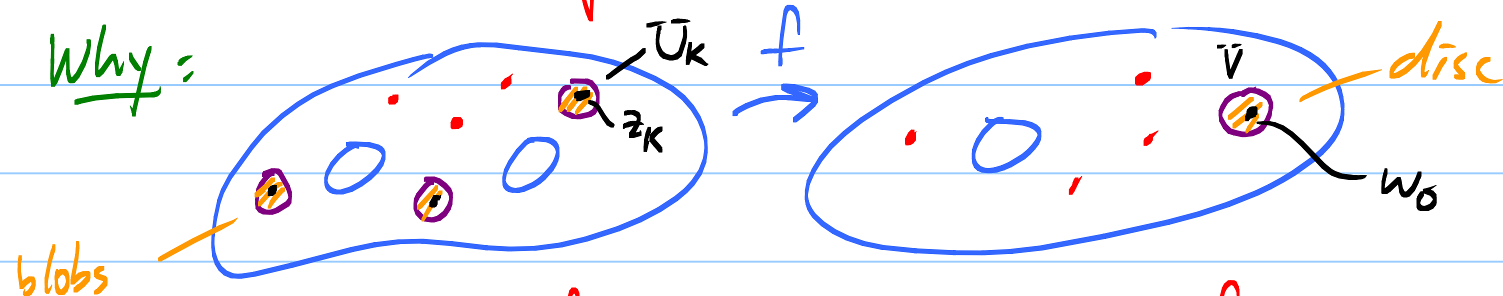
Consequently, $f \in C^\infty(\bar{\Omega}_1)$.

Pf: **Claim**: $\Lambda u := u \mapsto f' \cdot (u \circ f)$

$$\Lambda : L^2(\Omega_2) \to L^2(\Omega_1)$$

$$\|\Lambda\| = \sqrt{m}\,!$$

Why:



• = zeroes of f', discrete

Images of zeroes of f', discrete

$f: \text{blob} \xrightarrow[\text{onto}]{1-1} \text{disc}$. There are m blobs.

Recall: $|f'| = \left|\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\right|^2$.

Suppose $u \in C_0^\infty(\bar{V})$, ($\bar{V}$ = small disc about w_0)

$$f: \bar{U}_K \xrightarrow[\text{onto}]{1-1} \bar{V}$$

$$\underbrace{\sum \int_{U_K} |f'(u \circ f)|^2 \, dA}_{\int_\Omega |f'(u\circ f)|^2 \, dA} = m \int_{\bar{V}} |u|^2 \, dA$$

$$\|\Lambda f\|^2_{L^2(\Omega_1)} = m \|u\|^2_{L^2(\Omega_2)}$$

Avoid discrete set, use a partition of unity, use denseness in L^2, get

$$\Lambda : L^2(\Omega_2) \to L^2(\Omega_1), \quad \|\Lambda\| = \sqrt{m}.$$

<u>Next</u>: Given $u \in C^\infty(\bar{\Omega}_2)$, know

$$u \cong h + \frac{\partial \varphi}{\partial z} \quad \text{where} \quad h \in A^\infty(\Omega_2)$$
$$\text{and} \quad \varphi \in C_0^\infty(\bar{\Omega}_2)$$

and $h = P_2 u$. Now

$$P_1(\Lambda u) \cong P_1\Big(\underbrace{f'(h\circ f)}_{\text{analytic}} + \underbrace{f'\big(\tfrac{\partial \varphi}{\partial z}\circ f\big)}_{\frac{\partial}{\partial z}(\underbrace{\varphi\circ f}_{})} \Big)$$

$$= f'(h\circ f) + 0$$

$C_0^\infty(\Omega_1)$

$\perp H^2(\Omega_1)$

$$\boxed{P_1 \Lambda = \Lambda P_2} \quad \checkmark$$

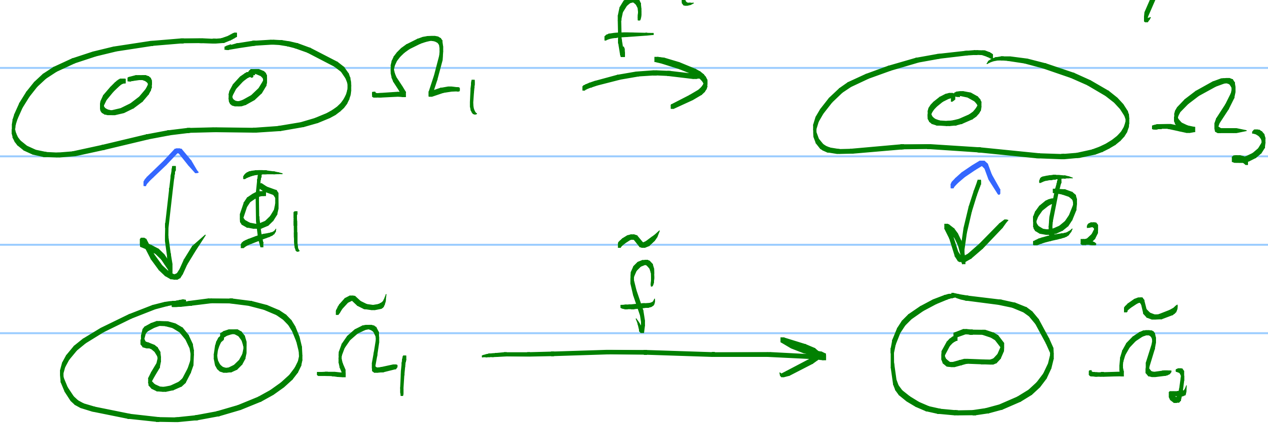
Alternate viewpoint: Bergman space $=$ L^2 space of analytic $(1,0)$-forms $h\,dz$.

$$\langle u\,dz, v\,dz\rangle = \frac{1}{i}\int u\bar{v}\, dz\wedge d\bar{z}$$

$\Lambda =$ differential d. $\boxed{P_1 d = d P_2}$

Next, show f is C^∞ up to $b\Omega_1$.

<u>Step 1</u>: Map Ω_1 and Ω_2 to domains with smooth real analytic boundary:



Know Φ_j smooth up to boundaries.

Replace $\tilde{f}$ by f.

Recall: $1 = P_2(1) = P_2(\underbrace{1 - \frac{\partial \varphi}{\partial \bar{z}}}_{\psi})$

where
1) φ is real analytic on nbhd of $b\Omega_2$
2) $\varphi\big|_{b\Omega_2} = 0$
3) $1 - \frac{\partial \varphi}{\partial \bar{z}} \in C_0^\infty(\Omega_2)$.

$$\text{Now}\quad f' = f'(1 \circ f) = f' \cdot \left[(P_2 \psi) \circ f \right]$$

$$= P_1\Big(\underbrace{f'(\psi \circ f)}_{C_0^\infty(\Omega_1)} \Big)$$

and we know P_1 of something with compact support extends analytically past the boundary in domains with real analytic bndry.

So f' extends, and so is in $C^\infty(\bar{\Omega}_1)$.

Hopf Lemma $\Rightarrow$ f' non-vanishing on $b\Omega_1$.

Remark: All this works in $\mathbb{C}^n \to \mathbb{C}^n$ maps! $\underbrace{\det[f']}_{u}$ in place of f'!

$\Lambda\varphi = u\cdot(\varphi\circ f)$. Bergman Proj transform as above. Get $u = \det[f']$ smooth up to $b\Omega_1$ if Bergman projection preserves $C^\infty(\bar{\Omega}_1)$. [Condition R.] Get $u \in C^\infty(\bar{\Omega}_1)$.

$u\cdot f_j = u(z_j \circ f)$ like before to see $u\cdot f_j \in C^\infty(\bar{\Omega}_1)$. $f_j = \frac{u f_j}{u}$

Ouch! u might vanish at points on $b\Omega_1$. In fact, it must if u has zeroes inside Ω_1.

Strategy: Show that u can vanish at most to finite order.

$u\cdot f_j^K = u(z_j^K \circ f) \in C^\infty(\bar{\Omega}_1)$ for all K.

Prove a Weierstraß Preparation Theorem up to boundary to factor and deduce $f_j \in C^\infty(\bar{\Omega}_1)$.