Zeroes of the Szegö Kernel

Recall transformation formula : $f : \Omega_1 \to \Omega_2$ biholo

$$S_1(z,w) = \sqrt{f'(z)}\, S_2(f(z), f(w))\, \overline{\sqrt{f'(w)}}$$

Properties of zeroes preserved under conformal maps. Can map a C^∞ smooth n-connected domain to one with real analytic boundary.

Facts: $S(z,w)$ is in $C^\infty((\bar\Omega \times \bar\Omega) - \{(z,z) : z \in b\Omega\})$

and $L(z,w)$ is in $C^\infty((\bar\Omega \times \bar\Omega) - \{(z,z) : z \in \bar\Omega\})$

When Ω has real analytic boundary, S and L extend past the boundary away from singular diagonal points.

Recall: Ahlfors map $f_a = \dfrac{S(z,a)}{L(z,a)}$ has

n zeroes — $(n-1)$ from zeroes of $S(\cdot,a)$ and 1 from the pole of $L(\cdot,a)$.

Thm: As $a \to A \in \gamma_k$, the zeroes of $S(\cdot,a)$ become distinct and simple and head off one to each of the other boundary curves γ_j, $j \neq k$.

Pf: $\boxed{\overline{S(z,a)} = \frac{1}{i} L(z,a) T(z)}$, $z \in b\Omega$
(A) $a \in \Omega$.

Still holds as $a \to A \in b\Omega$, $A \neq z$.

(A) with $z, a \in b\Omega$, $z \neq a$.

$$\boxed{\overline{S(a,z)} = \frac{1}{i} L(a,z) T(a)} = -\frac{1}{i} L(z,a) T(a)$$
(B)

Recall $S(a,z) = \overline{S(z,a)}$, $L(a,z) = -L(z,a)$.

$$\overline{S(z,a)} = \frac{1}{i} L(z,a) T(z) = \overline{-\frac{1}{i} L(z,a) T(a)} = S(a,z)$$

$$= \frac{1}{i} \overline{L(z,a)}\, \overline{T(a)}$$

(*) $\boxed{T(z) L(z,a) T(a) = \overline{L(z,a)}}$ $z, a \in b\Omega$
$z \neq a$.

Claim: Write $(a_1, a_2, \dots, a_{n-1})$ for the zeroes of $S(\cdot, a)$ counted with multiplicity.

These zeroes tend toward $b\Omega$ as $a \to A \in \gamma_k$.

Why: $f_a(z) = \frac{S(z,a)}{L(z,a)}$, Fix $z \in \Omega$.

If $a \in b\Omega$: $\overline{S(a,z)} = \frac{1}{i} L(a,z) T(a)$

Aha! $\left|\frac{S(z,a)}{L(z,a)}\right| = |T(a)| = 1.$

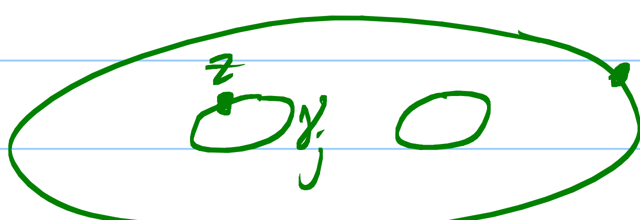
So $|f_a(z)| \to 1$ for fixed $z \in \Omega$

as $a \to A \in \gamma_K$. Zeroes get shoved out and toward boundary.

<u>Next</u>: Consider zeroes of $L(z,a)$ on bndry, $a \in \gamma_K$. <u>Claim</u>: $L(z,a)$ has <u>at least one</u> zero in z on each γ_j, $j \neq k$.

<u>why</u>: Suppose no zeroes on γ_j, $j \neq k$.

$$(*) \qquad T(z) L(z,a) T(a) = \overline{L(z,a)}.$$



Δ arg's using $(*)$ $\Bigg\}$ $\Delta_{\gamma_j} \arg T + \Delta_{\gamma_j} \arg L_a = -\Delta_{\gamma_j} \arg L_a$

$$-(\pm 2\pi) = 2\Delta_{\gamma_j} \arg L_a$$

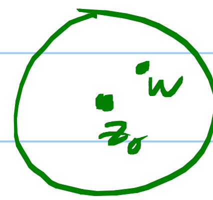
no zeroes

$= 2$ (multiple of 2π)

Not possible. Conclude $L(z,a)$ has at least one zero in z on each γ_j, $j \neq k$, if $a \in \gamma_k$.

<u>Fact</u>: When $a \in b\Omega$, zeroes of S_a and L_a coincide. Easy $S(z,a) = \frac{1}{i} \overline{L(z,a)}\, T(z)$.

<u>Mini.fact</u>: Zeroes of $S(z,a)$ in z are "continuous" in a.



zero of g in $D_\varepsilon(z_0)$ at w

$$\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{g'(z)}{g(z)}\, z\, dz = w^\ell$$ where

ℓ is mult. of zero.

$$\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{\frac{\partial}{\partial z} S(z,a)}{S(z,a)}\, z\, dz$$ is continuous in a.

In order to get at least one zero of S_a on each γ_j, zeroes must separate and become simple and head one each toward each γ_j, $j \neq k$.

<u>Interesting observation</u>: $R(z)$ = anti-holo reflection function, mapping inside Ω

to outside and $R(z) = z$ on $b\Omega$.

$\left[R(z) = \overline{S(z)} . \right]$ $a \in \Omega$ near $b\Omega$.

$b \in \Omega$ fixed.

$$\begin{cases} \overline{S(z,a)} = \frac{1}{i} L(z,a) T(z) \\ \overline{S(z,b)} = \frac{1}{i} L(z,b) T(z) \end{cases}$$

Divide: $\overline{\left(\frac{S(z,a)}{S(z,b)} \right)} = \frac{L(z,a)}{L(z,b)}$ for $z \in b\Omega$.

Put $R(z)$ in for z. Use $R(z) = z$ on $b\Omega$.

$$\overline{\left(\frac{S(R(z),a)}{S(R(z),b)} \right)} = \frac{L(z,a)}{L(z,b)} \quad \text{true for } z \in b\Omega$$

analytic in z <u>outside</u> Ω

analytic in z <u>inside</u> Ω

And they agree for $z \in b\Omega$.

Schwarz Reflection gives analytic extension.

<u>Conclude</u>: Since $L(z,a)$ has no zeroes inside near $b\Omega$, $S(z,a)$ has no zeroes outside near

$b\Omega$.

Repeat, with $R(z)$ on the L side.

Conclude: As zero of $S(z,a)$ heads toward γ_j as $a \to A \in \gamma_k$, there is a zero of $L(z,a)$ outside Ω that tends toward same point. $(\text{zero of } L_a) = R(\text{zero of } S_a)$

For more details, see Chap. 25.

Remark: Bergman kernel $K(z,a)$ has $n-1$ zeroes in z inside Ω when a is near the boundary. They don't tend toward the boundary. In fact $K(z,a) \neq 0$ if $z, a \in b\Omega$, $z \neq a$.