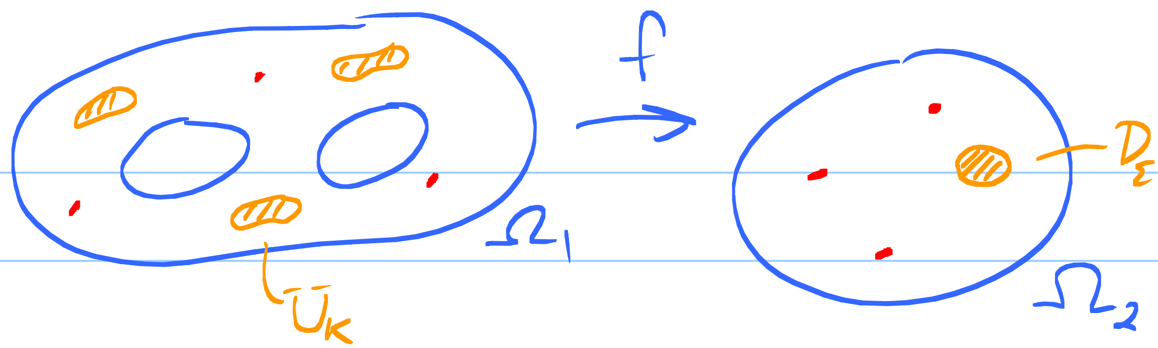

$f : \bar{U}_K \xrightarrow[\text{onto}]{1-1} D_\varepsilon \qquad K = 1, 2, \dots, m$

Let $F_K = f^{-1}$ on D_ε.

$$\Lambda_1 u = f'(u \circ f) \qquad \Lambda_1 : L^2(\Omega_2) \to L^2(\Omega_1)$$

$$\Lambda_2 v = \sum_{K=1}^{m} F_K'(v \circ F_K) \qquad \Lambda_2 : L^2(\Omega_1) \to L^2(\Omega_2)$$

Isolated sing are removable $\to H^2(\Omega_1) \to H^2(\Omega_2)$

L^2 Riemann Removable Singularity Theorem:

Suppose h analytic on $D_r(a) - \{a\}$. If $h \in L^2(D_r(a))$, then a is a removable singularity.

Facts: $\langle \Lambda_1 u, v \rangle = \langle u, \Lambda_2 v \rangle$

$$f'(z) K_2(f(z), w) = \sum_{K=1}^{m} K_1(z, F_K(w)) \overline{F_K'(w)}$$

Back to Ahlfors maps:

Lemma: The Ahlfors map f_a extends to the double as a meromorphic function.

Pf: Easy. Know $|f_a(z)| = 1$ when $z \in b\Omega$.

So $\boxed{f_a(z) = 1/\overline{f_a(z)}}$ for $z \in b\Omega$.

Theorem: Fix $a \in \Omega$. There is a point $b \in \Omega$ (and in fact, all but finitely many will work) such that f_a and f_b form a primitive pair for the field of meromorphic functions on the double.

Sketch of Pf: Know $f_a = \frac{S_a}{L_a}$, $f_b = \frac{S_b}{L_b}$.

Let $\tilde{f}_a$ and $\tilde{f}_b$ denote mero extensions to $\hat{\Omega}$. Need $\tilde{f}_b$ that separates $\tilde{f}_a^{-1}(\infty)$.

Tool: Linear span of $\{S(\cdot, b) : b \in \Omega\}$ is dense in $A^\infty(\Omega)$.

Corollary: The Bergman kernel $K(z,w)$ is equal to

$$f_a'(z)\, R\left(f_a(z), f_b(z), \overline{f_a(w)}, \overline{f_b(w)}\right) \overline{f_a'(w)}$$

where R is rational.

Remark: In the simply connected case

$$K(z,w) = \frac{f'(z)\overline{f'(w)}}{\pi\left(1-f(z)\overline{f(w)}\right)^2} \quad \text{where } f \text{ is a}$$

Riemann map.

Dirichlet Problem in multiply connected domains

$$S_a u = h + \bar{H}\,\bar{T}$$

$$u = \frac{h}{S_a} + \frac{\bar{H}\,\bar{T}}{S_a} = \frac{P(S_a u)}{S_a} + \frac{\overline{P(L_a \bar{u})}}{\overline{L_a}}$$

Simply Connected: (holo) + (anti-holo)
solves D' Prob.

Multiply Connected: (mero) + (anti-holo)

ouch! $n-1$ zeroes of S_a.

Pick $a \in \Omega$ so that the $n-1$ zeroes of S_a are distinct and simple.

Pick one point b_j inside γ_j. (One b_j per $n-1$ of holes.) Let $\psi_j = \ln|z-b_j|$

Observe: $\Delta_{\gamma_k}(\text{harmonic conj of } \psi_j) = \begin{cases} 0 & \text{if } k \neq j \\ \neq 0 & \text{if } k = j \end{cases}$

<u>Big idea</u>: Given $u \in C^\infty(b\Omega)$, want the harmonic extension to Ω. Replace u by $\mathcal{U} = u - \sum_{j=1}^{n-1} c_j \psi_j$ in formula

$$\mathcal{U} = \frac{P(S_a \mathcal{U})}{S_a} + \overline{\frac{P(L_a \overline{\mathcal{U}})}{L_a}}$$

Hmmm. Can we pick c_j's so that $P(S_a \mathcal{U})$ has zeroes at the $n-1$ zeroes on S_a.

<u>Linear algebra problem</u>: $\{a_1, a_2, \ldots, a_{n-1}\}$ zeroes of S_a.

$$P(S_a u)(a_k) = \sum_{j=1}^{n-1} c_j P(S_a \psi_j)(a_k) \qquad k = 1, \ldots, n-1$$

<u>Claim</u>: $\operatorname{Det}\left[P(S_a \psi_j)(a_k)\right] \neq 0.$

<u>Pf</u>: Suppose $\sum_{j=1}^{n-1} \tilde{c}_j P(S_a(\psi_j))(a_k) = 0$. Then

$$\underbrace{\sum_j \tilde{c}_j \psi_j}_{\tilde{u}} = \underbrace{\frac{P(S_a \tilde{u})}{S_a}} + \overline{\frac{P(L_a \bar{\tilde{u}})}{L_a}}$$

zeroes cancel! holomorphic at a_k's

$\sum \tilde{c}_j \psi_j$ has a harmonic conjugate!

No way! Just look at increase of the harmonic conjugate around a inner hole γ_j. Conclude all $\tilde{c}_j$ must be zero. So $\det \neq 0$.

Get sol^n to D. Prob

$$u = h + \bar{H} + \sum_{j=1}^{n-1} c_j \ln|z - b_j|$$

$$h = \frac{P(S_a \tilde{u})}{S_a}, \quad H = \frac{P(L_a \bar{\tilde{u}})}{L_a}$$

Thm: Given $u \in C^\infty(b\Omega)$, the sol^n to D. Prob is in $C^\infty(\bar{\Omega})$.

Similar theorem in real analytic category.

Important objects: w_j is sol^n to D. Prob with boundary data $\begin{cases} 1 & \text{on } \gamma_j \\ 0 & \text{on } \gamma_k, \ k \neq j. \end{cases}$

w_j are the harmonic measure functions.

$F_j' = 2 \frac{\partial w_j}{\partial z}$ is an analytic function on Ω.

Note: F_j' is locally the derivative of $(w_j + i v_j)$ where v_j is a local harmonic conjugate for w_j.

Fact: F_j' is not the derivative of a global analytic fcn on Ω.

Think: w_j is like $\ln|z-b_j|$

F_j' is like $\frac{1}{z-b_j}$