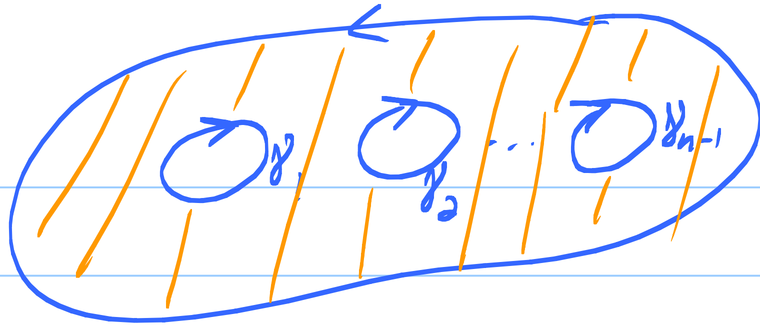


Ω n-connected

ω_j harmonic on Ω $= \begin{cases} 1 & \text{on } \gamma_j \\ 0 & \text{on } \gamma_k,\ k \neq j \end{cases}$

<u>Note</u>: $\sum_{j=1}^{n} \omega_j \equiv 1$ $(*)$

<u>Def^n</u>: $F_j' = 2\dfrac{\partial \omega_j}{\partial z}$ is analytic on Ω

$(*)$ shows that $F_n' = -\sum_{k=1}^{n-1} F_k$

<u>Thm</u>: V = complex linear span of $\{F_j\}_{j=1}^{n}$ is $n-1$ dim^l. $\{F_j\}_{j=1}^{n-1}$ form a basis.

<u>Pf</u>: See book, Chap on harmonic measure functions. Only uses complex Green's Identity.

<u>Remark</u>: $E \subset b\Omega$. Solve D. Prob. with boundary values 1 on E and 0 on $b\Omega - E$. Call it u_E. Pick $z_0 \in \Omega$. <u>Def^n</u>: $m(E) = u_E(z_0)$. is called "harmonic measure" in some books.

Def^n: $A_{ij} = \int_{\gamma_i} F_j' \, dz$

$A = [A_{ij}]_{i,j=1,\dots,n-1}$ is the matrix of periods.

Theorem: $\det \mathbb{A} \neq 0$.

Application: On n-connected Ω, given analytic h on Ω, $\exists$ constants c_j and analytic H such that

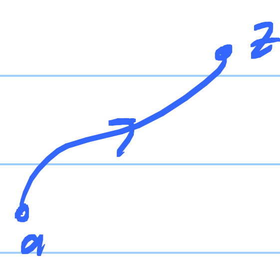
$$h = H' + \sum_{j=1}^{n-1} c_j F_j'$$

Pf: Think:

$$\left[\int_{\gamma_i} h \, dz \right] = \vec{0} + \mathbb{A} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}.$$

Solve for c_j's. Now $g = h - \sum c_j F_j'$ has zero periods about each γ_j, $j = 1, \dots, n-1$. General Cauchy Thm shows these γ_j form a homology basis for Ω. So $\int_\gamma g \, dz = 0$ for any closed γ in Ω. Now

$$H(z) = \int_{\gamma_a^z} g(\zeta) \, d\zeta$$



<u>Thm</u>: In simply connected domains, harmonic = real part of global analytic. In n-connected Ω, harmonic u is given by

$$u = h + \overline{H} + \sum_{j=1}^{n-1} c_j w_j \qquad (\dagger)$$

<u>Pf</u>: Think: Take $\frac{\partial}{\partial z}$ of $(\dagger)$:

$$\text{analytic} \rightarrow \frac{\partial u}{\partial z} = h' + 0 + \frac{1}{2}\sum_{j=1}^{n-1} c_j F_j'$$

Integrate around γ_i

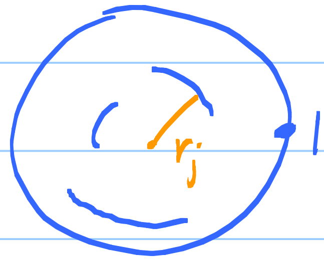
$$\left[\int_{\gamma_j} \frac{\partial u}{\partial z}\, dz\right] = 0 + \frac{1}{2} A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

solve for c_j's. <u>Pf</u>: $\frac{\partial u}{\partial z} - \sum_{j=1}^{n-1} c_j F_j'$ is analytic with zero periods. So $= h'$.

$$\frac{\partial u}{\partial z} = h' + \sum c_j F_j'$$

Now, $u - h - 2\sum c_j w_j$ is anti-analytic $\overline{H}$.

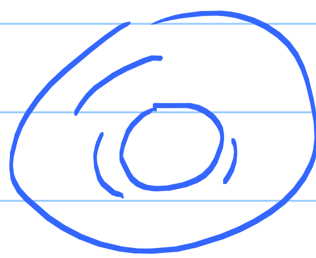
Fact: n-connected Ω is biholomorphic to

1) $f : \Omega \to$ 

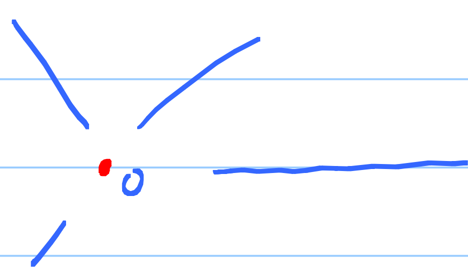
$n-1$ circular non-touching sub-arcs centered at 0

$$\operatorname{Ln}|f| = \sum_{j=1}^{n-1} (\operatorname{Ln} r_j)\, w_j$$

$f = e^{\operatorname{Ln}|f| + iv}$ Big idea: cook harmonic conjugate v so that f is single valued.

2) or 

3) or $\mathbb{C}$ minus n horizontal line segments
(vertical)

4) or 

5) Moonja Jeong: Ω is biholomorphic to

$$\left| z + \sum \frac{c_j}{z - b_j} \right| < 1 \longleftarrow$$ algebraic kernel functions