Duality of $A^\infty(\Omega)$ and $A^{-\infty}(\Omega)$.

Def^n: $A^\infty(\Omega) = C^\infty(\bar{\Omega}) \cap \{\text{holo fcns}\}$

$A^{-\infty}(\Omega) = \{g : g \text{ holo on } \Omega \text{ and}, \exists C, N$
$|g(z)| \le C\,\mathrm{dist}(z, b\Omega)^{-N}\}$

Theorem: The L^2 inner product on Ω extends to $A^\infty(\Omega) \times A^{-\infty}(\Omega)$.

How: Suppose $g, h \in A^\infty(\Omega)$.

Recall, given $N \in \mathbb{N}$, $\exists\ \varphi \in C^\infty(\bar{\Omega})$ with $\varphi|_{b\Omega} = 0$ and $h - \frac{\partial \varphi}{\partial \bar{z}} = \theta(z)\rho(z)^N$ where $\theta \in C^\infty(\bar{\Omega})$ and $\Omega = \{\rho < 0\}$, ρ is a defining function for Ω.

Recall: $\frac{\partial \varphi}{\partial \bar{z}} \perp H^2(\Omega)$. So

$$\langle h, g \rangle = \langle \underbrace{h - \tfrac{\partial \varphi}{\partial \bar{z}}}_{\in C_0^N(\Omega)}, g \rangle$$

can take seq $g \to G \in A^{-\infty}$.

Theorem: A^∞ and $A^{-\infty}$ are mutally dual.

1) A continuous linear fcnl on A^∞ is given by $h \mapsto \langle h, g \rangle$ for a

unique $g \in A^{-\infty}$.

2) Similar for cont. lin. funcl on $A^{-\infty}$.

Operator: $h \mapsto \underbrace{h^{-\frac{\partial\varphi}{\partial z}}}_{\Phi^N(h)}$ ← defines a diff operator with coeff in $C^\infty(\bar{\Omega})$.

Properties: $B \circ \Phi^N = B$ B = Berg. Proj.

$\Phi^N : W^s(\Omega) \to W_0^s(\Omega)$ for each s.

Fact: In $\mathbb{C}^n$ on a C^∞ smooth pseudoconvex domain Ω the A^∞, $A^{-\infty}$ duality is equivalent to Condition R: $B : C^\infty(\bar{\Omega}) \circlearrowleft$.

Composition Operator: $\varphi : D_1(0) \to D_1(0)$ analytic

Composition operator C_φ on the Hardy Space $H^2(bD_1(0))$ is defined via

$$(C_\varphi h)(z) = h(\varphi(z)).$$

[or $(C_{\varphi,\psi} h)(z) = \psi(z)\, h(\varphi(z))$]

Adjoint of C_φ: $\langle C_\varphi h, g \rangle = \langle h, C_\varphi^* g \rangle$.

Hmmm. Let $g = S_a \leftarrow$ Szegő kernel $S(z,a)$

$$\underbrace{(C_\varphi h)(a)}_{h(\varphi(a))} = \langle C_\varphi h, S_a \rangle = \langle h, C_\varphi^* S_a \rangle$$

$\Vert \leftarrow$ must be $=$

$$h(\varphi(a)) = \langle h, S_{\varphi(a)} \rangle$$

Fact: $C_\varphi^* S_a = S_{\varphi(a)}$

Idea: Approximate h by functions in the linear span of $\{S_a : a \in \Omega\}$ to compute $C_\varphi^* h$.

Think about constructive proof of density lemma

Key: $S_a = P C_a$

P ↑ Szegő proj; C_a ↑ Cauchy kernel $\frac{-1}{2\pi i} \frac{\overline{T(z)}}{\overline{z - a}}$

Key: $u = \mathcal{C}u + \frac{1}{2\pi i} \iint_\Omega \frac{\frac{\partial u}{\partial \bar w}}{w - z} \, dw \wedge d\bar w$

$$\bar{u} = \overline{\mathcal{C}u} - \frac{1}{2\pi i}\iint_{\Omega} \frac{\frac{\partial \bar{u}}{\partial w}}{\bar{w}-\bar{z}}\, d\bar{w}\wedge dw$$

Multiply by $\overline{T(z)}$.

$$\bar{u}\,\bar{T} = \underbrace{\overline{\mathcal{C}u}\,\bar{T}}_{\substack{\bar{H}\bar{T} \\ \perp H^2(\Omega)}} + \underbrace{\frac{1}{2\pi i}\iint_{\Omega} \frac{T(z)}{\bar{w}-\bar{z}}\cdot\frac{\partial \bar{u}}{\partial w}\, dw\wedge d\bar{w}}_{\text{Riemann sum of } C_w\text{'s.}}$$

Take Szegö proj:

$$(*)\quad \underbrace{P(\bar{u}\,\bar{T})}_{} = P\left(\frac{1}{2\pi i}\iint_{\Omega} \frac{T(z)}{\bar{w}-\bar{z}}\left(\frac{\partial \bar{u}}{\partial w} - \underset{\perp H^2(\Omega)}{\frac{\partial \Psi}{\partial w}}\right) dw\wedge d\bar{w}\right)$$

want $= h$.

$$\bar{u}\,\bar{T} = h$$

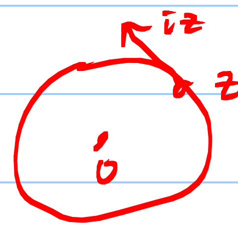
$$\bar{u} = hT$$

$$u = \bar{h}\,\bar{T}$$

$$\boxed{u = -i\,\bar{h}\,\bar{z}}$$

But on $D_1(0)$, we know

$$T(z) = iz$$



$$\frac{\partial \bar{u}}{\partial w} = i\,(h + wh')$$

$$(*)\quad h = P\left(\frac{1}{2\pi i}\iint_{\Omega} \frac{T(z)}{\bar{w}-\bar{z}}\, \Phi^{S}(h+wh')\, dw\wedge d\bar{w}\right)$$

$$h \cong P\left(\sum c_i C_{a_i} \right)$$

$$= \sum c_i S_{a_i} \cong \iint S_w \Phi^5 (h + wh')\, dA$$

Now $C_\varphi^* h \cong \sum c_i S_{\varphi(a_i)}$

$$(C_\varphi^* h)(z) = \iint_\Omega (h + wh') \overline{S(\varphi(w), z)}\, dw \wedge d\bar{w}$$

$$= \langle h + wh', S_z \circ \varphi \rangle_{H^2(\Omega)}$$