Dirichlet Problem on double quadrature domains

Ω a bounded C^∞ smooth n-connected double q. domain.

Facts: Then Ω has a real analytic boundary and the Schwarz function $S(z)$ extends mero to $\hat{\Omega}$, and T is the restriction to $b\Omega$ of a mero fcn on $\hat{\Omega}$ without poles on $b\Omega$.

$$T(z) = \begin{cases} h(z) \\ \overline{H(z)} \end{cases} \quad \text{for } z \in b\Omega$$

where h, H are mero on Ω, extend C^∞ to $b\Omega$.

Review: $\quad \overline{S_a} = \frac{1}{i} L_a T \quad$ and $L_a(z) = L(z,a) = \frac{1}{2\pi} \cdot \frac{1}{(z-a)} + \ell(z,a)$

$$\overline{S(z,a)} = \frac{1}{i} L(z,a)\, T$$

analytic in a

Diff wrt a: $\quad \overline{S^m(z,a)} = \frac{1}{i} L^m(z,a)\, T$

Fact: On a boundary arc-length q. domain,

$$\sum c_{jm} h^{(m)}(a_j) = \int_{b\Omega} h \, ds \quad \forall h$$

$$\Longrightarrow \boxed{\sum c_{jm} S_{a_j}^{(m)} \equiv 1} \ (*)$$

<u>Lemma</u>: $S_a^{(m)}$ and $L_a^{(m)}$ extend mero to $\hat{\Omega}$.

<u>Pf</u>: $(*)$ shows that

$$1 = \sum c_{jm} S_{a_j}^{(m)} = \overline{\frac{1}{i} \sum \overline{c_{jm}} L_{a_j}^{(m)} T}$$

To see that $S_a^{(n)}$ extends:

$$S_a^{(n)} = \frac{S_a^{(n)}}{1} = \overline{\frac{\frac{1}{i} L_a^{(n)} T}{\frac{1}{i} \sum \overline{c_{jm}} L_{a_j}^{(m)} T}} = \overline{H}$$

$\uparrow$ antiholo

on $b\Omega$. So $S_a^{(n)}$ extends.

Argument for $L_a^{(n)}$ similar.

<u>Fact</u>: $\begin{cases} z & \text{on } \Omega \\ \overline{S(z)} & \text{on } \tilde{\Omega} \end{cases}$ and $\begin{cases} S(z) & \text{on } \overline{\Omega} \\ \overline{z} & \text{on } \tilde{\Omega} \end{cases}$

form a primitive pair for $\hat{\Omega}$.

Conclude $S_a^{(n)} = R(z, S(z))$ on Ω <u>and</u>

$$S_a^{(n)} = R(z, \bar{z}) \text{ on } b\Omega.$$

Theorem: Partial fractions decomposition for a rational function $R(z,\bar{z})$ on the boundary of a double q. domain:

$$R(z,\bar{z}) = \underbrace{\left(\sum_{j,m} c_{jm} S_{a_j}^{(m)}\right) + \left(\sum_{i,n} b_{in} L_{a_i}^{(n)}\right)}_{\substack{\text{finite sums.} \\ \text{uniquely determined.}}}$$

Pf: Step 1: $R(z,\bar{z}) = R(z, S(z))$ extends R to Ω as a mero fcn.

Step 2: $\overline{S(z,a)} = \frac{1}{i}\left[\frac{1}{2\pi}\frac{1}{(z-a)} + \ell(z,a)\right]T$

diff wrt a: $\overline{S^{(m)}(z,a)} = \frac{1}{i}\underbrace{\left[\frac{1}{2\pi}\frac{m!}{(z-a)^{m+1}} + \ell^{(m)}(z,a)\right]}_{L^{(m)}(z,a)}T$

Step 3: Can subtract off linear combo of $L^{(m)}(z,a_j)$ to eliminate poles of $R(z,S(z))$.

Get $h(z) = R(z,S(z)) - \sum c_{jm} L_{a_j}^{(m)}$

that is rational on $b\Omega$, and extends analytically to Ω.

<u>Step 4</u>: Claim: h = linear combo of $S_{a_i}^{(n)}$.

<u>Pf</u>: $$\langle h, g\rangle = \int_{b\Omega} h(z)\,\overline{g(z)}\,ds =$$

$h(z) = Q(z,\bar z)$ on $b\Omega$, Q rational

$$= \int_{b\Omega} Q(z,\bar z)\,\overline{g(z)}\,\underbrace{ds}_{T\,d\bar z \,=\, \bar H\, d\bar z}$$

$$= \int_{b\Omega} Q\left(\overline{S(z)}, \bar z\right)\overline{g(z)}\,\bar H\,d\bar z$$

$$= 2\pi i \sum (\text{Residues in } \Omega)$$

Residue Thm.

$$= \text{linear combo of } g^{(n)}(a_i)$$

Must conclude h = linear combo of $S_{a_i}^{(n)}$.

Done: $$R(z, S(z)) = \left(\sum c_{jm} S_{a_j}^{(m)}\right) + \left(\sum b_{in} L_{a_i}^{(n)}\right)$$

<u>Remark</u>: Found b_{in}'s by subtracting off poles.
Can find c_{jm}'s in a similar manner:

$$\underbrace{\overline{R(z,S(z))}}\ \overline{T} = \sum \bar{c}_{jm} \underbrace{\overline{S_{a_j}^{(m)}}}_{L_{a_j}^{(m)}}\overline{T} + \left(\sum \bar{b}_{in} \underbrace{\overline{L_{a_i}^{(n)}}}_{S_{a_i}^{(n)}}\overline{T}\right)$$

$$\underbrace{\overline{R(\overline{S(z)},\bar{z})}}_{\text{mero}}\ \overline{H}$$

Just match up poles.

<u>Theorem</u>: Solving D. Prob with rational data on a double q. domain is as easy as partial fractions.

<u>Pf</u>: Assume Ω is simply connected.

$$S_a u = h + \overline{H}\,\overline{T} \leftarrow \text{orthog decomp}$$

$$S_a \underset{\substack{\uparrow \\ \text{rational}}}{u} = \left(\sum c_{jm} S_{a_j}^{(m)}\right) + \left(\sum b_{in} L_{a_i}^{(n)}\right)$$

$$u = \sum c_{jm} \underset{\substack{\uparrow \\ \text{nice}}}{\frac{S_{a_j}^{(m)}}{S_a}} + \sum b_{in} \underbrace{\frac{L_{a_i}^{(n)}}{S_a}} = \underset{\substack{\uparrow \\ \text{nice}}}{\frac{\overline{S_{a_i}^{(n)}}}{\overline{L_a}}}$$

$$u = \left(\begin{matrix}\text{nice analytic} \\ \text{extends to } \hat{\Omega}\end{matrix}\right) + \overline{\left(\text{same}\right)}$$

Theorem: On a double q. domain, the Dirichlet to Neumann map takes rational functions to rational fcns.

Def^{n}: Boundary data u. Extend harmonically to U on Ω.

$V = \frac{\partial U}{\partial n}$, the normal derivative on $b\Omega$.

D. to N. map: $u \longmapsto V$.