The Green's Function and Bergman Kernel of an n-Connected Domain

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We will establish the identity relating the Bergman kernel $K(z,w)$ and Green's function $G(z,w)$ of an n-connected domain Ω with C^∞ boundary:

$$K(z,w) = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial \overline{w}}. \tag{1}$$

We first recall that the Green's function for such a domain Ω is given by

$$G(z,w) = -\log|z-w| + \psi(z,w), \tag{2}$$

where for each $w \in \Omega$ the function $\psi(z,w)$ satisfies

$$\Delta_z \psi(z,w) = 0, \quad \forall\ z \in \Omega,$$

$$\psi(z,w) = \log|z-w|\,, \quad \forall\ z \in \partial\Omega.$$

Equation (1) is established in [1] as Theorem 31.3 on page 180. Our proof here will make use of the following:

Theorem 1. *Let Ω be a bounded domain in $\mathbb{C}$ such that $\partial\Omega$ consists of n disjoint C^∞ Jordan curves. Let $w \in \Omega$ and let $\varphi(\cdot, w)$ be the solution to the following Dirichlet problem:*

$$\Delta_z \varphi(z,w) = 0 \quad \forall\ z \in \Omega,$$

$$\varphi(z,w) = \frac{1}{z-w}, \quad \forall\ z \in \partial\Omega.$$

Then the Bergman kernel $K(z,w)$ for Ω is given by

$$K(z,w) = \frac{1}{\pi}\frac{\partial \overline{\varphi}(z,w)}{\partial z}.$$

Theorem 1 is established in [1] as Theorem 25.4 on page 138. The main idea is to use the complex Green's identity for integrals and the density of $A^\infty(\Omega)$ in the Bergman space $H^2(\Omega)$. Similar ideas are used in Theorem 31.3 to prove equation (1), but the argument is a bit more tricky. The motivation for our proof of equation (1) is to avoid the tricky argument by realizing that Theorem 1 contains most of what we need.

The remainder of what we need is that the functions $G(z,w)$ and $\psi(z,w)$ are C^∞ functions of *both* variables z, w wherever they are defined. Assuming this, we prove our theorem.

Theorem 2. *Let Ω be a bounded domain in $\mathbb{C}$ such that $\partial\Omega$ consists of n disjoint C^∞ Jordan curves. The Bergman kernel $K(z,w)$ and the Green's function $G(z,w)$ satisfy*

$$K(z,w) = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial \overline{w}}.$$

Proof. We compute the $\overline{w}$ derivative of equation (2) to obtain

$$\frac{\partial G(z,w)}{\partial \overline{w}} = -\frac{1}{2(\overline{w}-\overline{z})} + \frac{\partial \psi(z,w)}{\partial \overline{w}}. \tag{3}$$

Differentiating with respect to the z variable now gives

$$\frac{\partial^2 G(z,w)}{\partial z \partial \overline{w}} = \frac{\partial^2 \psi(z,w)}{\partial z \partial \overline{w}}. \tag{4}$$

Our assumption on the differentiability of ψ implies that

$$\Delta_z\left(\frac{\partial \psi(z,w)}{\partial \overline{w}}\right) = 4\frac{\partial^2}{\partial \overline{z}\partial z}\left(\frac{\partial \psi(z,w)}{\partial \overline{w}}\right)$$

$$= 4\frac{\partial}{\partial \overline{w}}\left(\frac{\partial^2}{\partial \overline{z}\partial z}\psi(z,w)\right) = \frac{\partial}{\partial \overline{w}}\left(\Delta_z \psi(z,w)\right) = 0, \quad \forall\ z \in \Omega.$$

Furthermore, equation (1) implies that for each $w \in \Omega$ we have $G(z,w) = 0$ for all $z \in \partial\Omega$. Hence, for each $w \in \Omega$ it follows that $\dfrac{\partial G(z,w)}{\partial \overline{w}} = 0$ for all $z \in \partial\Omega$. Equation 3 now implies that

$$\frac{\partial \psi(z,w)}{\partial \overline{w}} = \frac{1}{2(\overline{w}-\overline{z})} = -\frac{1}{2(\overline{z}-\overline{w})}, \quad \forall\ z \in \partial\Omega.$$

Now, the function $\varphi(z,w) = -2\overline{\left(\dfrac{\partial \psi(z,w)}{\partial \overline{w}}\right)}$ satisfies

$$\Delta_z \varphi(z,w) = 0 \quad \forall\ z \in \Omega,$$

$$\varphi(z,w) = \frac{1}{z-w}, \quad \forall\ z \in \partial\Omega.$$

Therefore, by Theorem 1 and equation (4) we obtain

$$K(z,w) = \frac{1}{\pi}\frac{\partial \overline{\varphi}(z,w)}{\partial z} = -\frac{2}{\pi}\frac{\partial^2 \psi(z,w)}{\partial z \partial \overline{w}} = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial \overline{w}}.$$

$\square$

References

[1] Bell, S. R. *The Cauchy Transform, Potential Theory, and Conformal Mapping*, 2^{nd} ed. CRC Press, 2016.