Steven R. Bell, Purdue University

The Cauchy Transform, Potential Theory, and Conformal Mapping

Contents

Preface

The Cauchy integral formula is an old and beautiful result. Anybody with an undergraduate mathematics education knows something about this formula. Since the Cauchy integral has been studied for so long by so many people, it is tempting to believe that everything there is to know about the integral is known. Such beliefs are always wrong. In 1978, Norberto Kerzman and Elias M. Stein [K-S] discovered a very basic, yet previously unknown, property of the Cauchy integral. They discovered that the Cauchy integral is *nearly self adjoint*. This discovery constituted a shift in the bedrock of complex analysis. The vibrations were felt immediately in the areas of boundary behavior of holomorphic functions, Hardy spaces, and mapping problems. I hope this book will help dissipate the shock wave through some areas that were thought to be peacefully settled forever.

Students encountering complex analysis for the first time are delighted to see how many of the basic theorems follow directly from the Cauchy integral formula. In the first course on complex variables, the Cauchy integral is applied only to holomorphic functions. This book is intended to be part of a second course on complex variables. In it, the Cauchy integral will be applied to nonholomorphic functions. Since the Cauchy integral of a nonholomorphic function does not reproduce that function, this process gives rise to a transform. The Cauchy transform maps functions defined on the boundary of a domain to holomorphic functions on the domain.

The Kerzman-Stein breakthrough allows many of the basic objects of a second course in complex analysis to be described in terms of the Cauchy transform. More theorems than ever before can be seen to be direct corollaries of simple facts about the Cauchy integral. In this book, we will deduce the Riemann mapping theorem, we will solve the Dirichlet and Neumann problems for the Laplace operator, we will construct the Poisson kernel, and we will solve the inhomogeneous Cauchy-Riemann equations. We will do all this in a very constructive way using formulas stemming from the Kerzman-Stein theorem.

Kerzman and Stein made their discovery while studying the Szegő kernel in several complex variables. Discussion of the Szegő projection and kernel have always been considered to be too advanced to include in a second course in complex analysis. However, because of the Kerzman-Stein result, we will be able to discuss these objects in very simple and concrete terms. We will also be able to study the Bergman kernel. These classical objects are very useful in the study of conformal mappings and we explore many of their applications.

Another motivation for writing this book has been that the new formulas expressing the classical objects of potential theory and conformal mapping in terms of Cauchy transforms and Szegő projections have led to new numerical methods for computing these objects. In the past, it has been difficult for nonspecialists to find accessible descriptions of the background material needed to understand the new formulas. A reader who is interested only in understanding the Kerzman-Stein numerical method for computing conformal mappings may read Chapters 1–8, then skip to Chapter 28.

Many classical problems in conformal mapping and potential theory on finitely connected domains in the plane can be reduced by means of the Riemann mapping theorem to problems on domains whose boundaries consist of finitely many real analytic curves. In this book, we study many problems for which this standard reduction is applicable, however, the full force of the real analyticity of the boundary curves is only rarely needed. In most situations we will encounter, it will suffice to study problems on a domain whose boundary curves are merely C^∞ smooth. This is why we fixate on problems of C^∞ behavior of objects on domains with C^∞ smooth boundaries. Although many complex analysts may look upon such a fixation as being unwholesome, I believe it is justified in the context of this book. I should also mention that many results that we prove in the C^∞ category can be quite routinely generalized to apply under milder smoothness hypotheses. I did not feel compelled to explain these more general results because they are irrelevant to the main theme of the book.

I have written this book thinking that my readers have completed a one-semester course on complex analysis and that they know some rudimentary facts about real analysis and Hilbert space. The facts from Hilbert space and measure theory are so few and so basic that they could easily be learned along the way. I also assume that my reader knows about C^∞ partitions of unity and a little about differential forms, enough to understand Green's Theorem in planar domains. Any voids in the reader's background can be filled by consulting Ahlfors [Ah], Rudin [Ru1, Ru2], and Spivak [Sp].

I wrote the first edition of this book thinking of it as a tour of a beautiful region of classical analysis viewed through the new lenses of the Kerzman-Stein theorem. Writing the first edition propelled me to study quadrature domains, complexity in complex analysis, and improvements upon the Riemann mapping theorem. I have added four new chapters to this second edition: Chapters 22 and 23 on quadrature domains and Chapters 33 and 34 on complexity of the objects of complex analysis and improved Riemann mapping theorems. This new material brings the tour to a destination where I currently wander.

I would like to thank David Barrett, Harold Boas, Young-Bok Chung, Anthony Thomas, and Alan Legg for reading preliminary drafts of this book and for making many constructive suggestions. I'd like to thank Steve Krantz for planting the idea of writing this book in my head.

West Lafayette, April 2015

Table of symbols

$(\partial/\partial z)$	the ∂-operator	4
$(\partial/\partial\bar{z})$	the $\bar{\partial}$-operator	4
φ_z	$= \partial\varphi/\partial z$	6
$\varphi_{\bar{z}}$	$= \partial\varphi/\partial\bar{z}$	6
$\langle u, v\rangle_b$	inner product on $L^2(b\Omega)$	11
$\langle u, v\rangle_\Omega$	inner product on $L^2(\Omega)$	69
$\langle h, g\rangle$	extension of $L^2(\Omega)$ inner product to $A^\infty(\Omega) \times A^{-\infty}(\Omega)$	166
$\lVert u\rVert$	the norm on $L^2(b\Omega)$ (or $L^2(\Omega)$)	11 (69)
$\lVert u\rVert_s$	the norm on $C^s(b\Omega)$ (or $C^s(\Omega)$)	35 (165)
$\lVert u\rVert_{-s}$	the norm on $A^{-s}(\Omega)$	165
$\lVert u\rVert_s^A$	the C^s norm on an arc $A \subset b\Omega$	141
$\int$	integral on a curve in the plane	5
$\int\int$	integral on an open set in the plane	5
P.V.$\int$	Principal Value integral	17
$\Delta\arg h$	increase in argument of h around $b\Omega$	40, 60
ρ	boundary defining function	6
ω_k	harmonic measure function	97
$[A_{jk}]$	matrix of periods	100
$\mathcal{A}$	Kerzman-Stein operator	13
$A(z, w)$	Kerzman-Stein kernel	14, 19
$A^\infty(\Omega)$	holomorphic functions in $C^\infty(\overline{\Omega})$	9
$A^\infty(b\Omega)$	$\equiv A^\infty(\Omega)$	13
$A^{-\infty}(\Omega)$	the dual of $A^\infty(\Omega)$	165
$A^{-s}(\Omega)$	a subspace of $A^{-\infty}(\Omega)$	165
$b\Omega$	boundary of Ω	3
B	the Bergman projection	70
$\mathcal{C}$	Cauchy transform	9
$\mathcal{C}^*$	adjoint of Cauchy transform	12
$C_a(z)$	kernel of Cauchy transform	27
$C^k(\overline{\Omega})$	functions with k continuous derivatives on $\overline{\Omega}$	3
$C^\infty(\overline{\Omega})$	$= \cap_{k=0}^\infty C^k(\overline{\Omega})$	3
$C^\infty(b\Omega)$	C^∞ smooth functions on $b\Omega$	3
$C^\omega(b\Omega)$	real analytic functions on real analytic $b\Omega$	48
$C^s(b\Omega)$	functions on $b\Omega$ with s continuous derivatives	35

$D_r(a)$	disc of radius r about a	5
$\mathcal{E}$	Poisson extension operator	45
F'_k	$= 2\partial\omega_k/\partial z$	97
$\mathcal{F}'$	the linear span of the functions F'_k	98
G	the classical Green's operator	73
$G(z,a)$	the Green's function	177
$G_a(z)$	$= (1/2\pi)(z-a)^{-1}$	30
$\mathcal{H}$	the Hilbert transform	131 (17)
$H^2(b\Omega)$	Hardy space	13
$H^2(b\Omega)^\perp$	orthogonal complement of $H^2(b\Omega)$ in $L^2(b\Omega)$	15
$H^2(\Omega)$	the Bergman space	69
$H^2(\Omega)^\perp$	the orthogonal complement of $H^2(\Omega)$ in $L^2(\Omega)$	74
$K_a(z) = K(z,a)$	the Bergman kernel function	70
$L^2(b\Omega)$	square integrable functions on $b\Omega$	11
$L^2(\Omega)$	square integrable functions on Ω	69
$L_a(z) = L(z,a)$	Garabedian's kernel	29
$\ell(z,w)$	$= L(z,w) - (2\pi)^{-1}(z-w)^{-1}$	145
P	the Szegő projection	13
$P^\perp$	the projection onto $H^2(b\Omega)^\perp$	16
P.V.	Principal Value	17
$S(z)$	the Schwarz function	118
$S_a(z) = S(z,a)$	the Szegő kernel	27
T	the complex unit tangent vector function	3

1

Introduction

One reason that the unit disc $D_1(0)$ in the complex plane is the most pleasant place on earth to do function theory is that the monomials z^n are orthogonal in two basic inner product spaces associated to the disc; they are orthogonal with respect to the boundary arc length inner product

$$\langle u, v\rangle_b = \int u(e^{i\theta})\overline{v(e^{i\theta})}\, d\theta$$

and with respect to the area measure inner product

$$\langle u, v\rangle = \iint_{D_1(0)} u\,\bar{v}\, dx\, dy.$$

Therefore, after the theory of Hilbert space had been developed, it was inevitable that the spaces obtained by forming the Hilbert space completions of the polynomials with respect to the boundary and the area inner products would be studied. These spaces turned out to be as heavenly as the unit disc itself. On these two spaces, classical complex analysis, measure theory, and functional analysis blend together naturally to yield scores of theorems that any mathematician would find appealing.

The completion of the polynomials with respect to the boundary inner product is the Hardy space, and the completion with respect to the area inner product is the Bergman space. These spaces have assumed prominent spots in the literature of graduate mathematics. The purpose of this book is to define and study the analogous spaces for domains in the plane more general than the disc and to reap some of the remarkable applications that these spaces have found in potential theory and conformal mapping. Although the subject matter of this book has not been considered elementary enough to become a standard part of the graduate mathematics curriculum, I hope to present it in such a way that it will be seen that it could easily be included in a second course on complex variables. Because this is part of my mission, and because it required very little extra effort and ink, I have tried to make this book comprehensible to a first year graduate student in mathematics.

The paper of Kerzman and Stein [K-S] is at the foundation of the present work; it is the Kerzman-Stein viewpoint that allows the classical results I will describe to be simplified to the point that they can be understood by beginners in complex analysis. Although this book is primarily expository, it does have a research component. It is research to look at old results from a new point of view, one that allows proofs to be streamlined and simplified. New outlooks always give rise to new theorems. I have added the last two chapters of this new second edition of the book to illustrate this point.

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The improved Cauchy integral formula

In this book, we will study functions defined inside and on the boundary of a bounded domain Ω in the plane with C^∞ smooth boundary. Such a domain is finitely connected and its boundary consists of finitely many C^∞ smooth simple closed curves. We let $b\Omega$ denote the boundary of Ω. If Ω is n-connected, there are C^∞ complex valued functions $z_j(t)$ of $t \in [0,1] \subset \mathbb{R}$, $j = 1, \ldots n$, that parameterize the n boundary curves of Ω in the *standard sense*. This means that, if $z_j(t)$ parameterizes the j-th boundary curve of Ω, it is understood that $z_j(t)$ and all its derivatives agree at the endpoints $t = 0$ and $t = 1$, that $z_j'(t)$ is nowhere vanishing, and that $z_j(t)$ traces out the curve exactly once. Furthermore, $-iz_j'(t)$ is a complex number representing the direction of the *outward* pointing normal vector to the boundary at the point $z_j(t)$. We say that a function g defined on the boundary of Ω is C^∞ smooth on $b\Omega$ if, for each j, $g(z_j(t))$ is a C^∞ function of t on $[0,1]$, all of whose derivatives agree at the endpoints 0 and 1. This definition seems to depend on the choice of the parameterization functions $z_j(t)$, but it is an easy exercise to see that it in fact does not. We let $C^\infty(b\Omega)$ denote the space of C^∞ functions on $b\Omega$. One of the most important C^∞ smooth functions on the boundary of Ω is the *complex unit tangent function* T. If $z \in b\Omega$, then $T(z)$ is equal to the complex number of unit modulus that represents the direction of the tangent vector to $b\Omega$ at z pointing in the direction of the standard orientation of the boundary. To be precise, T is characterized by the formula $T(z_j(t)) = z_j'(t)/|z_j'(t)|$. Since the differential dz is given by $dz = z_j'(t)\,dt$ and the differential ds of arc length is given by $ds = |z_j'(t)|\,dt$, we see that $dz = T\,ds$.

If k is a positive integer, $C^k(\overline{\Omega})$ denotes the space of continuous complex valued functions on $\overline{\Omega}$ whose partial derivatives up to and including order k exist and are continuous on Ω and extend continuously to $\overline{\Omega}$. The space $C^\infty(\overline{\Omega})$ is the set of functions in $C^k(\overline{\Omega})$ for all k.

Everyone learns that a holomorphic function can be represented by its Cauchy integral. Everyone should also learn that even nonholomorphic functions have Cauchy integral representations. *In this book, unless explicitly stated otherwise, Ω will denote a bounded domain in the plane with C^∞ smooth boundary.*

Theorem 2.1. *If u is a function in $C^1(\overline{\Omega})$, then*

$$u(z) = \frac{1}{2\pi i}\int_{b\Omega}\frac{u(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\iint_{\Omega}\frac{\partial u/\partial\bar{\zeta}}{\zeta - z}\,d\zeta\wedge d\bar{\zeta}$$

for all $z \in \Omega$.

To make this book self-contained, we will spend a moment to define the notation used in this theorem before starting its proof. The differential of a complex valued function $u(z)$, considered as a function of $(x, y) \in \mathbb{R}^2$ (via $z = x + iy$), is a differential one-form given by $du = (\partial u/\partial x)dx + (\partial u/\partial y)dy$. If we define complex valued one-forms $dz = dx + idy$ and $d\bar{z} = dx - i\,dy$, then, after some linear algebra, we may express du as $F\,dz + G\,d\bar{z}$ where F is *defined* to be $\partial u/\partial z$ and G is *defined* to be $\partial u/\partial\bar{z}$. The result of this linear algebra gives

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$$
$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right).$$

Note that, by the Cauchy-Riemann equations, u is holomorphic if and only if $(\partial u/\partial\bar{z}) = 0$; this is why the $\bar{\partial}$-operator $\partial/\partial\bar{z}$ is so important in complex analysis. Also note that if u is holomorphic, then the formula in Theorem 2.1 reduces to the classical Cauchy integral formula.

The operators $\partial/\partial z$ and $\partial/\partial\bar{z}$ will be used on almost every page of this book. The following properties of these operators will be used routinely. The complex conjugate of $\partial u/\partial z$ is $\partial\bar{u}/\partial\bar{z}$ and the complex conjugate of $\partial u/\partial\bar{z}$ is $\partial\bar{u}/\partial z$. Suppose that $u(z)$ is defined as a composition of functions via $u(z) = g(w)$ where $w = f(z)$. The complex chain rule can be written

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial u}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial z}, \text{ and}$$
$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial \bar{z}} + \frac{\partial u}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial \bar{z}}.$$

These formulas can be verified by simply writing out both sides in terms of real derivatives, or preferably, they can be deduced by using the chain rule for differential forms and the definition of dz and $d\bar{z}$ given above.

Before we can begin the proof of Theorem 2.1, we need to explain a complex version of integration by parts that is analogous to the Green's identities of classical analysis. Let ζ denote a complex variable. Since $d[u\,d\zeta] = du\wedge d\zeta$, we may use elementary properties of differential forms to see that,

$$d[u\,d\zeta] = \frac{\partial u}{\partial\zeta}\,d\zeta\wedge d\zeta + \frac{\partial u}{\partial\bar{\zeta}}\,d\bar{\zeta}\wedge d\zeta = \frac{\partial u}{\partial\bar{\zeta}}\,d\bar{\zeta}\wedge d\zeta.$$

We may use this fact in an application of Stokes' Theorem to deduce that

$$\int_{b\Omega} u\, d\zeta = \iint_{\Omega} \frac{\partial u}{\partial \bar{\zeta}}\, \partial\bar{\zeta} \wedge \partial\zeta.$$

Although it may be old fashioned, we like to write $\iint$ for the integral over Ω and $\int$ for the integral over the lower dimensional $b\Omega$. This last result will be used repeatedly in this book. The complex conjugate of the identity is also important:

$$\int_{b\Omega} u\, d\bar{\zeta} = \iint_{\Omega} \frac{\partial u}{\partial \zeta}\, \partial\zeta \wedge \partial\bar{\zeta}.$$

We will call these two identities *complex Green's identities.* Notice that, because $d\zeta = dx + i\, dy$ and $d\bar{\zeta} = dx - i\, dy$, it follows that $\partial\zeta \wedge \partial\bar{\zeta} = -2i\, dx\, dy$, and hence the double integral in the formula represents an integral with respect to Lebesgue area measure.

Proof of Theorem 2.1. Let $D_\epsilon(z)$ denote the disc of radius ϵ about the point z. To prove the Cauchy formula, let Ω_ϵ denote the domain $\Omega - D_\epsilon(z)$ for small $\epsilon > 0$. Apply the complex Green's identity on the domain Ω_ϵ using the function $U(\zeta) = u(\zeta)/(\zeta - z)$. Note that $\partial U/\partial\bar{\zeta} = (\partial u/\partial\bar{\zeta})/(\zeta - z)$. Thus, if we parameterize the boundary of $D_\epsilon(z)$ in the counterclockwise sense, we may write

$$\int_{b\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta - \int_{bD_\epsilon(z)} \frac{u(\zeta)}{\zeta - z}\, d\zeta = \iint_{\Omega_\epsilon} \frac{\partial u/\partial\bar{\zeta}}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta.$$

We now let $\epsilon \to 0$. Because u is continuous, the integral over the boundary of $D_\epsilon(z)$ tends to $2\pi i\, u(z)$. We now claim that the integrand in the double integral is in L^1. To see this, use polar coordinates centered at z in order to write $d\bar{\zeta} \wedge d\zeta = 2i\, dx\, dy = 2ir\, dr\, d\theta$ and $|\zeta - z|^{-1} d\bar{\zeta} \wedge d\zeta = 2i\, dr\, d\theta$. Finally, because the integrand is in $L^1(\Omega)$, the Cauchy formula follows. □

The Dirichlet problem for the Laplace operator is a very important tool in the study of harmonic functions. To study holomorphic functions, we will need to understand solutions to the $\bar{\partial}$-equation, that is, solutions to $\partial u/\partial\bar{z} = v$.

Theorem 2.2. *Suppose that $v \in C^\infty(\overline{\Omega})$. Then the function u defined via*

$$u(z) = \frac{1}{2\pi i} \iint_{\Omega} \frac{v(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

satisfies $\partial u/\partial\bar{z} = v$ and $u \in C^\infty(\overline{\Omega})$.

Since the complex conjugate of $\partial u/\partial \bar{z}$ is equal to $\partial \bar{u}/\partial z$, Theorem 2.2 implies that the equation $\partial u/\partial z = v$ can also be solved with $u \in C^\infty(\overline{\Omega})$ via an integral formula similar to the one in the statement of the theorem.

To prove Theorem 2.2, we will need the following lemma. The lemma is somewhat technical in nature. The reader can be assured that the effort spent on its proof will be fully rewarded later when we will use it to give short and easy proofs of some classical theorems in analysis.

Before we can state the lemma, we need to define some terminology and mention some elementary facts. A function will be said to vanish to order m on the boundary of Ω if it, together with its partial derivatives (with respect to x and y) up to and including order m, vanish on $b\Omega$. Two functions are said to agree to order m on the boundary of Ω if their difference vanishes to order m on $b\Omega$. Note that to say that two functions agree to order *zero* on the boundary simply means that they are equal on the boundary.

Because the boundary of Ω is C^∞ smooth, there is a real valued function ρ that is C^∞ smooth on a neighborhood of $\overline{\Omega}$ with the property that $\Omega = \{\rho < 0\}$, $b\Omega = \{\rho = 0\}$, and $d\rho \neq 0$ on $b\Omega$. Such a function is called a *defining function* for Ω. We will indicate briefly how to construct such a ρ, leaving the details to the reader. The implicit function theorem implies that, given a sufficiently small arc Γ in the boundary of Ω, it is possible to find a C^∞ diffeomorphism defined on a neighborhood of Γ that maps Γ one-to-one onto a segment in the real axis of the complex plane. A suitable local defining function for Ω near Γ can be obtained by pulling back the function $\pm\text{Im } z$ via the diffeomorphism. A global ρ can be constructed from a finite number of local ones by means of a partition of unity.

We will need to know that a function ψ in $C^\infty(\overline{\Omega})$ vanishes to order m on $b\Omega$ if and only if $\psi = \theta\rho^{m+1}$ for some function $\theta \in C^\infty(\overline{\Omega})$. This fact is easy to understand in case Ω is the upper half plane and $\rho(z) = -\text{Im } z$. In this case, the fact follows from Taylor's formula with an explicit integral as the remainder term. To understand the general case, use the local diffeomorphisms described above to reduce the question to the easy case. We can now state and prove the lemma.

Lemma 2.1. *Suppose that* $v \in C^\infty(\overline{\Omega})$*. Then, for each positive integer* m*, there exists a function* $\Phi_m \in C^\infty(\overline{\Omega})$ *that vanishes on the boundary of* Ω *such that* $\partial\Phi_m/\partial\bar{z}$ *and* v *agree to order* m *on* $b\Omega$*.*

Proof of the Lemma. Let ρ be a defining function for Ω. We will construct the functions Φ_m inductively. A subscript $\bar{z}$ will denote differentiation with respect to $\bar{z}$. Let χ be a C^∞ function on $\mathbb{C}$ that is equal to one on a small neighborhood of $b\Omega$ and that vanishes on a neighborhood of the zero set of $\rho_{\bar{z}}$ (that agrees with the set where $d\rho = 0$). First, we will construct a function Φ_0 that satisfies the conclusion of the lemma for $m = 0$ by setting $\Phi_0 = \theta_0\rho$ and then choosing θ_0 appropriately. Since

$$(\theta_0\rho)_{\bar z} = (\theta_0)_{\bar z}\rho + \theta_0\rho_{\bar z},$$

it is clear that it would be a good idea to set $\theta_0 = v/\rho_{\bar z}$. This would yield that $(\theta_0\rho)_{\bar z} = (\theta_0)_{\bar z}\rho + v$ and hence, that $(\theta_0\rho)_{\bar z} - v$ vanishes on $b\Omega$. The only problem with doing this is that $\rho_{\bar z}$ vanishes at some points in Ω. This is where the function χ comes in. By setting $\theta_0 = \chi v/\rho_{\bar z}$, we obtain a function Φ_0 such that $(\Phi_0)_{\bar z}$ is equal to v on $b\Omega$. In fact, $(\Phi_0)_{\bar z} - v = \Psi_0\rho$ where

$$\Psi_0 = (\chi - 1)v\rho^{-1} + \frac{\partial}{\partial\bar z}\left(\chi v/\rho_{\bar z}\right),$$

and it is clear that Ψ_0 is in $C^\infty(\overline{\Omega})$.

Now suppose that we have constructed functions Φ_k for $k < m$ that satisfy the conclusion of the lemma. Then $(\Phi_{m-1})_{\bar z} - v = \Psi_{m-1}\rho^m$ for some $\Psi_{m-1} \in C^\infty(\overline{\Omega})$. We will complete the induction process by setting $\Phi_m = \Phi_{m-1} - \theta_m\rho^{m+1}$ and by choosing θ_m appropriately. Now

$$\begin{aligned}(\Phi_m)_{\bar z} - v &= (\Phi_{m-1})_{\bar z} - v - (\theta_m\rho^{m+1})_{\bar z} \\ &= \Psi_{m-1}\rho^m - (\theta_m)_{\bar z}\rho^{m+1} - \theta_m(m+1)\rho^m\rho_{\bar z}.\end{aligned}$$

The term in this expression containing ρ^{m+1} vanishes to order m on $b\Omega$ and can be ignored. We would like to choose θ_m so that the terms containing ρ^m cancel out near $b\Omega$. This can easily be done by taking $\theta_m = \chi\Psi_{m-1}/((m+1)\rho_{\bar z})$. With this choice, we have established that $(\Phi_m)_{\bar z} - v = \Psi_m\rho^{m+1}$ for a function $\Psi_m \in C^\infty(\overline{\Omega})$. This completes the induction. □

Proof of Theorem 2.2. We will prove the theorem by showing that u is in $C^m(\overline{\Omega})$ for each positive integer m. Given a positive integer m, let Φ denote the function Φ_m furnished by the lemma and let $\Psi = v - \Phi_{\bar z}$ be the corresponding function that vanishes to order m on $b\Omega$. By Theorem 2.1,

$$\Phi(z) = \frac{1}{2\pi i}\iint_\Omega \frac{\partial\Phi/\partial\bar\zeta}{\zeta - z}\, d\zeta \wedge d\bar\zeta.$$

Subtracting this equation from the definition of u yields that

$$u(z) - \Phi(z) = \frac{1}{2\pi i}\iint_\Omega \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta.$$

Now, we may consider Ψ to be a C^m function on all of $\mathbb{C}$ by extending it to be equal to zero on $\mathbb{C} - \Omega$. We may perform a simple change of variables in the last integral to obtain

$$u(z) - \Phi(z) = -\frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\Psi(z-\zeta)}{\zeta} \, d\zeta \wedge d\bar{\zeta}. \tag{2.1}$$

Because $1/\zeta$ is locally in L^1 and because Ψ has compact support, it is permissible to differentiate m times under the integral sign to see that $u - \Phi$ is in $C^m(\overline{\Omega})$; hence, so is u. Since this is true for each positive integer m, it follows that u is in $C^\infty(\overline{\Omega})$.

Finally, we must show that $\partial u/\partial\bar{z} = v$. Suppose $z_0 \in \Omega$ and let χ be a function in $C_0^\infty(\Omega)$ that is one on a neighborhood of z_0. Now

$$u(z) = \frac{1}{2\pi i} \iint_{\Omega} \frac{\chi v}{\zeta - z} \, d\zeta \wedge d\bar{\zeta} + \frac{1}{2\pi i} \iint_{\Omega} \frac{(1-\chi)v}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

Since $(1 - \chi)$ vanishes near z_0, we may differentiate under the second integral when z is close to z_0 to see that the second integral is holomorphic in z near z_0. Hence, $\partial u/\partial\bar{z}$ is equal to $\partial/\partial\bar{z}$ of the first integral for z near z_0. Since χv has compact support, we may treat the first integral as an integral over $\mathbb{C}$ and we may change variables and differentiate under the integral as we did above to obtain

$$\frac{\partial u}{\partial \bar{z}}(z) = -\frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\left(\partial(\chi v)/\partial\bar{\zeta}\right)(z-\zeta)}{\zeta} \, d\zeta \wedge d\bar{\zeta}$$

for z near z_0. We may further manipulate this integral by reversing the change of variables and by using Theorem 2.1. We obtain

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2\pi i} \iint_{\Omega} \frac{\left(\partial(\chi v)/\partial\bar{\zeta}\right)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta} = \chi v$$

for z near z_0. Hence, $\partial u/\partial\bar{z} = v$ near z_0 and the proof is complete. □

3

The Cauchy transform

If u is a C^∞ function defined on the boundary of a bounded domain Ω in the plane with C^∞ smooth boundary, then the Cauchy transform of u is a holomorphic function $\mathcal{C}u$ on Ω given by

$$(\mathcal{C}u)(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{u(\zeta)}{\zeta - z}\,d\zeta.$$

In the study of harmonic functions, the Poisson integral plays a very important role; the Poisson integral establishes a one-to-one correspondence between continuous functions on the boundary and harmonic functions on the interior that assume those functions as boundary values. In complex analysis, the Cauchy transform plays a similar part; however, the interaction is more subtle because not all functions on the boundary can be the boundary values of a holomorphic function. In this chapter, we will spell out some basic properties of the Cauchy transform.

Let $A^\infty(\Omega)$ denote the space of holomorphic functions on Ω that are in $C^\infty(\overline{\Omega})$.

Theorem 3.1. *The Cauchy transform maps $C^\infty(b\Omega)$ into $A^\infty(\Omega)$.*

Proof. Let $u \in C^\infty(b\Omega)$ and let U be a function in $C^\infty(\overline{\Omega})$ that is equal to u on $b\Omega$. Theorem 2.1 allows us to write

$$U = \mathcal{C}u + \frac{1}{2\pi i}\iint_{\Omega} \frac{U_{\bar\zeta}(\zeta)}{\zeta - z}\,d\zeta \wedge d\bar\zeta,$$

and it follows from Theorem 2.2 that the function defined by the double integral is in $C^\infty(\overline{\Omega})$. The theorem is proved. □

Theorem 3.1 allows us to view the Cauchy transform as an operator that maps the space $C^\infty(b\Omega)$ into $C^\infty(\overline{\Omega})$, or even as an operator from $C^\infty(b\Omega)$ into itself. Theorem 3.1 also implies the following result.

Theorem 3.2. *Suppose that h is a holomorphic function on Ω that extends to be a continuous function on $\overline{\Omega}$. If the boundary values of h are in $C^\infty(b\Omega)$, then $h \in C^\infty(\overline{\Omega})$.*

The analogue of this theorem for harmonic functions is also true, but it is considerably more difficult to prove. It is remarkable that Theorem 3.2 can be proved so easily.

It is quite easy to deduce a local version of Theorem 3.2. Suppose Γ is an open arc in the boundary of Ω, and suppose that h is a holomorphic function on Ω that extends continuously to $\overline{\Omega}$ such that the boundary values of h on Γ are C^∞ smooth. Let z_0 be a point in Γ and let χ be a real valued function in $C^\infty(b\Omega)$ with compact support in Γ that is equal to one on a small neighborhood of z_0. Now $\mathcal{C}h = \mathcal{C}(\chi h) + \mathcal{C}[(1-\chi)h]$. Since χh is in $C^\infty(b\Omega)$, the term $\mathcal{C}(\chi h)$ is in $A^\infty(\Omega)$. Since $(1-\chi)h$ is zero in a neighborhood of z_0, a quick glance at the formula for the Cauchy transform reveals that, in fact, the term $\mathcal{C}[(1-\chi)h]$ extends holomorphically past the boundary near z_0. Thus, we have proved the following theorem.

Theorem 3.3. *Suppose that h is a holomorphic function on Ω that extends to be a continuous function on $\overline{\Omega}$. If the boundary values of h are C^∞ smooth on an open arc Γ in $b\Omega$, then all the derivatives of h extend continuously from Ω to $\Omega \cup \Gamma$.*

The proofs of Theorems 2.2 and 3.1 contain a constructive method for computing the boundary values of the Cauchy transform of a smooth function. Since we will continue to use the notation set up in those proofs in the remainder of this chapter, it is worth summarizing the key ingredients in the construction in the form of the next theorem. Let $u \in C^\infty(b\Omega)$ and let U be a function in $C^\infty(\overline{\Omega})$ that is equal to u on $b\Omega$. For a given positive integer m, let Φ be a function furnished by Lemma 2.1 such that $U_{\bar{z}} - \Phi_{\bar{z}}$ vanishes to order m on $b\Omega$ and Φ vanishes on $b\Omega$. We may now apply the Cauchy integral formula of Theorem 2.1 to obtain

$$U - \Phi = \mathcal{C}u + \frac{1}{2\pi i}\iint_\Omega \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

where $\Psi = U_{\bar{z}} - \Phi_{\bar{z}}$. Since Ψ can be viewed as a function in $C^m(\mathbb{C})$ via extension by zero, we may change variables and differentiate under the integral m times as we did in the proof of Theorem 2.2 to see that $\mathcal{C}u$ and its derivatives up to order m extend continuously to the boundary. Since Φ vanishes on the boundary and since $U = u$ on the boundary, we may express the boundary values of the Cauchy integral as follows.

Theorem 3.4. *Suppose that $u \in C^\infty(b\Omega)$. If m is a positive integer, there is a function $\Psi \in C^\infty(\overline{\Omega})$ that vanishes to order m on the boundary such that the boundary values of $\mathcal{C}u$ are expressed via*

$$(\mathcal{C}u)(z) = u(z) - \frac{1}{2\pi i}\iint_\Omega \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}, \qquad z \in b\Omega.$$