One of the goals of this book is to define and study the space of functions on the boundary of Ω that arise as the L^2 boundary values of holomorphic functions on Ω. To do this, we will need to study the L^2 adjoint of the Cauchy transform.

Let ds denote the differential element of arc length on the boundary of Ω and let $z_j(t)$, $j = 1, \ldots, n$, denote functions that parameterize the n boundary curves of Ω. For u and v in $C^\infty(b\Omega)$, the L^2 inner product on $b\Omega$ of u and v is defined via $\langle u, v\rangle_b = \int_{b\Omega} u\,\overline{v}\,ds$. The space $L^2(b\Omega)$ is defined to be the Hilbert space obtained by completing the space $C^\infty(b\Omega)$ with respect to this inner product. It is not hard to see that $L^2(b\Omega)$ is equal to the set of complex valued functions u on $b\Omega$ such that $u(z_j(t))$ is a measurable function of t for each j and $\|u\|^2 = \sum_{j=1}^n \int_0^1 |u(z_j(t))|^2 |z_j'(t)|\,dt$ is finite, and that this definition is independent of the choice of the parameterization of the boundary.

Suppose that u and v are in $C^\infty(b\Omega)$. We know that $\mathcal{C}u$ is also in $C^\infty(b\Omega)$. We wish to construct a function $\mathcal{C}^*v$ in $C^\infty(b\Omega)$ that satisfies $\langle \mathcal{C}u, v\rangle_b = \langle u, \mathcal{C}^*v\rangle_b$ for all $u \in C^\infty(b\Omega)$. Although we have used the notation $\mathcal{C}^*$, and we think of $\mathcal{C}^*$ as being the *adjoint* of $\mathcal{C}$, we must emphasize that, for the time being, we must refer to $\mathcal{C}^*$ as the *formal adjoint* of $\mathcal{C}$; this is because the crucial property $\langle \mathcal{C}u, v\rangle_b = \langle u, \mathcal{C}^*v\rangle_b$ will only be shown to hold for u and v in $C^\infty(b\Omega)$, a space that is not a Hilbert space. Later, we will see that $\mathcal{C}^*$ agrees with the L^2 adjoint of $\mathcal{C}$.

Given a positive integer m, we may express the boundary values of $\mathcal{C}u$ as described in Theorem 3.4. For $z \in b\Omega$, let us write $(\mathcal{C}u)(z) = u(z) - \mathcal{I}(z)$ where we define

$$\mathcal{I}(z) = \frac{1}{2\pi i} \iint_\Omega \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}. \tag{3.1}$$

Now, since $m \geq 1$, the function Ψ vanishes to at least order one on the boundary, and hence $\Psi = \theta\rho^2$ for some defining function ρ and $\theta \in C^\infty(\overline{\Omega})$. It follows that the integrand in the double integral $\mathcal{I}$ is a continuous function of (ζ, z) on $\overline{\Omega} \times b\Omega$. This will allow us to change the order of integration (via Fubini's theorem) when we write out the expression for $\langle \mathcal{C}u, v\rangle_b$.

Note that since the unit tangent vector function T is unimodular, it follows that $\overline{T} = 1/T$. Since $dz = T\,ds$, it also follows that $ds = \overline{T}\,dz$. Now,

$$\langle \mathcal{C}u, v\rangle_b = \langle u - \mathcal{I}, v\rangle_b,$$

and writing out the term $\langle \mathcal{I}, v\rangle_b$ and using Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{z\in b\Omega}\left(\frac{1}{2\pi i}\iint_{\zeta\in\Omega}\frac{\Psi(\zeta)}{\zeta - z}\, d\zeta\wedge d\bar{\zeta}\right)\,\overline{v(z)}\, ds\\
&\quad = \iint_{\zeta\in\Omega}\Psi(\zeta)\left(\frac{1}{2\pi i}\int_{z\in b\Omega}\frac{\overline{v(z)}}{\zeta - z}\, ds\right)\, d\zeta\wedge d\bar{\zeta}\\
&\quad = \iint_{\Omega}\Psi\; \mathcal{C}(-\bar{v}\,\overline{T})\; d\zeta\wedge d\bar{\zeta}.
\end{aligned}$$

The function $\mathcal{C}(-\bar{v}\,\overline{T})$ is in $A^\infty(\Omega)$. Remember that Ψ was constructed to be equal to $(\partial/\partial\bar{\zeta})(U-\Phi)$ where $\Phi = 0$ on $b\Omega$. Thus, we may further manipulate the last integral by applying the complex Green's formula to get

$$\iint_\Omega \frac{\partial}{\partial\bar{\zeta}}\left[(U-\Phi)\mathcal{C}(-\bar{v}\,\overline{T})\right]\;(-d\bar{\zeta}\wedge d\zeta) = \int_{b\Omega}(U-\Phi)\mathcal{C}(\bar{v}\,\overline{T})\; d\zeta = \langle u, V\rangle_b$$

where V is the complex conjugate of $T\mathcal{C}(\bar{v}\,\overline{T})$. If we now insert this expression for $\langle \mathcal{I}, v\rangle_b$ into $\langle \mathcal{C}u, v\rangle_b = \langle u - \mathcal{I}, v\rangle_b$, we get $\langle \mathcal{C}u, v\rangle_b = \langle u, \mathcal{C}^* v\rangle_b$ where

$$\mathcal{C}^* v = v - \overline{T}\,\overline{\mathcal{C}(\bar{v}\,\overline{T})}. \tag{3.2}$$

For now, $\mathcal{C}^* v$ is only defined when $v \in C^\infty(b\Omega)$. In the next chapter we will extend the definition to $L^2(b\Omega)$. Notice that whereas $\mathcal{C}$ could be viewed either as an operator that maps $C^\infty(b\Omega)$ into $C^\infty(\overline{\Omega})$ or into $C^\infty(b\Omega)$, $\mathcal{C}^*$ can only be viewed as an operator from $C^\infty(b\Omega)$ to itself.

4

The Hardy space, the Szegő projection, and the Kerzman-Stein formula

Let $A^\infty(b\Omega)$ denote the set of functions on $b\Omega$ that are the boundary values of functions in $A^\infty(\Omega)$. The Hardy space, $H^2(b\Omega)$, is defined to be the closure in $L^2(b\Omega)$ of $A^\infty(b\Omega)$. (This is not the standard definition of the Hardy space given in, say Stein [St], however, we will explain later why it is equivalent to any of the more standard definitions.) To streamline our notation, we will identify the space $A^\infty(\Omega)$ with $A^\infty(b\Omega)$.

Because $H^2(b\Omega)$ is a closed subspace of $L^2(b\Omega)$, we may consider the orthogonal projection P of $L^2(b\Omega)$ onto $H^2(b\Omega)$. This projection is called the *Szegő projection*. At the moment, we know that P is a bounded operator on $L^2(b\Omega)$ and that the Cauchy transform maps $C^\infty(b\Omega)$ into itself. We would also like to know that the Cauchy transform satisfies an L^2 estimate and that the Szegő projection maps $C^\infty(b\Omega)$ into itself. The key to deducing what we would like to know from what we now know is the Kerzman-Stein formula, which relates the Szegő projection to the Cauchy transform.

The Kerzman-Stein formula is

$$P(I + \mathcal{A}) = \mathcal{C} \tag{4.1}$$

where I denotes the identity operator and $\mathcal{A}$ is the Kerzman-Stein operator defined to be equal to $(\mathcal{C} - \mathcal{C}^*)$. Since we have only defined the operator $\mathcal{C}^*$ on $C^\infty(b\Omega)$, we will first establish the truth of this formula when it acts on functions in $C^\infty(b\Omega)$. Later, we will see that it is valid on $L^2(b\Omega)$. Because this formula plays a central role in this book, we will prove it in two different ways. The first proof rests on the observation that functions of the form $\overline{HT}$ where $H \in H^2(b\Omega)$ are orthogonal to $H^2(b\Omega)$. To see this, let h and H be functions in $H^2(b\Omega)$, and let h_i and H_i be sequences of functions in $A^\infty(\Omega)$ that converge in $L^2(b\Omega)$ to h and H, respectively. Now

$$\langle h, \overline{HT} \rangle_b = \lim_{i\to\infty} \langle h_i, \overline{H_i T} \rangle_b = \lim_{i\to\infty} \int_{b\Omega} h_i H_i \, dz = 0$$

by Cauchy's theorem. Hence $\overline{HT} \perp H^2(b\Omega)$. Now if $u \in C^\infty(b\Omega)$, then

$$[I + (\mathcal{C} - \mathcal{C}^*)]u = u + \mathcal{C}u - (u - \overline{T}\overline{\mathcal{C}(\overline{uT})}) = \mathcal{C}u + \overline{T}\overline{\mathcal{C}(\overline{uT})}.$$

The term $\mathcal{C}u$ is in $A^\infty(\Omega)$ and the term $\overline{T}\overline{\mathcal{C}(\overline{uT})}$ is of the form $\overline{TH}$ with $H \in A^\infty(\Omega)$, and is therefore orthogonal to $H^2(b\Omega)$. It follows that $P[I + (\mathcal{C} - \mathcal{C}^*)]u = \mathcal{C}u$, and this is the Kerzman-Stein formula.

The second proof uses the following general fact about projections: If $\langle u, H\rangle_b = \langle v, H\rangle_b$ for all functions H in a dense subspace of $H^2(b\Omega)$, then $Pu = Pv$. Let $u \in C^\infty(b\Omega)$ and $H \in A^\infty(\Omega)$. Now, using the fact that $\mathcal{C}H = H$, it follows that

$$\langle [I + (\mathcal{C} - \mathcal{C}^*)]u, H\rangle_b = \langle u, [I - (\mathcal{C} - \mathcal{C}^*)]H\rangle_b = \langle u, \mathcal{C}^* H\rangle_b = \langle \mathcal{C}u, H\rangle_b.$$

Since this identity holds for all $H \in A^\infty(\Omega)$ (which is a dense subspace of $H^2(b\Omega)$), we conclude that $P[I + (\mathcal{C} - \mathcal{C}^*)]u = P(\mathcal{C}u) = \mathcal{C}u$.

At first glance, it is not apparent why the Kerzman-Stein formula should be useful. Its utility stems from the fact that the Kerzman-Stein operator $\mathcal{A} = \mathcal{C} - \mathcal{C}^*$ is a much better operator than either $\mathcal{C}$ or $\mathcal{C}^*$. We will prove that $\mathcal{A}$ can be written

$$(\mathcal{A}u)(z) = \int_{\zeta \in b\Omega} A(z, \zeta)\, u(\zeta)\, ds$$

where $A(z, \zeta)$ is infinitely differentiable as a function of $(z, \zeta) \in b\Omega \times b\Omega$. What this means is that *the Cauchy transform is very close to being self-adjoint.* Let us assume this point for the moment and deduce some of its consequences. The most important consequence is that $\mathcal{A}$ is an operator that maps $L^2(b\Omega)$ into $C^\infty(b\Omega)$ and that satisfies an L^2 estimate $\|\mathcal{A}u\| \le c\|u\|$. Hence, because $\|Pu\| \le \|u\|$ for any $u \in L^2(b\Omega)$, the Kerzman-Stein formula yields that the Cauchy transform satisfies the L^2 estimate $\|\mathcal{C}u\| \le (1 + c)\|u\|$ for $u \in C^\infty(b\Omega)$. Since $C^\infty(b\Omega)$ is dense in $L^2(b\Omega)$, and since $\mathcal{C}$ maps $C^\infty(b\Omega)$ into $A^\infty(\Omega) \subset H^2(b\Omega)$, we have proved the following theorem.

Theorem 4.1. *The Cauchy transform extends to be a bounded operator from $L^2(b\Omega)$ into $H^2(b\Omega)$.*

We use the same symbol $\mathcal{C}$ to denote the extension of $\mathcal{C}$ to $L^2(b\Omega)$ as defined above. In fact, from this point on, unless stated otherwise, we will consider $\mathcal{C}$ to be defined as this operator on $L^2(b\Omega)$.

It follows from Theorem 4.1 that formula (3.2)

$$\mathcal{C}^* v = v - \overline{T}\,\overline{\mathcal{C}(\bar{v}\,\overline{T})}$$

expressing $\mathcal{C}^*$ in terms of $\mathcal{C}$ extends by the density of $C^\infty(b\Omega)$ in $L^2(b\Omega)$

to define a bounded operator (which we also denote by $\mathcal{C}^*$) from $L^2(b\Omega)$ to itself. Since the identity $\langle \mathcal{C}u, v\rangle_b = \langle u, \mathcal{C}^* v\rangle_b$ holds when u and v are in $C^\infty(b\Omega)$, and since $C^\infty(b\Omega)$ is dense in $L^2(b\Omega)$, this identity holds for u and v in $L^2(b\Omega)$. This shows that $\mathcal{C}^*$ is the L^2 adjoint of $\mathcal{C}$ and that we are no longer abusing notation.

We proved the Kerzman-Stein formula $P[I + (\mathcal{C} - \mathcal{C}^*)] = \mathcal{C}$ when it operates on $C^\infty(b\Omega)$. We now see that this same formula is also valid on $L^2(b\Omega)$ when the operators $\mathcal{C}$ and $\mathcal{C}^*$ are understood as operators on $L^2(b\Omega)$.

We are now in a position to take the L^2 adjoint of the Kerzman-Stein identity. Using the facts that $\mathcal{A}^* = (\mathcal{C}-\mathcal{C}^*)^* = -\mathcal{A}$, that $(P\mathcal{A})^* = \mathcal{A}^* P^*$, and that $P^* = P$ (because P is a projection), we obtain

$$(I - \mathcal{A})P = \mathcal{C}^*. \tag{4.2}$$

If we subtract this formula from the Kerzman-Stein formula, we get

$$P\mathcal{A} + \mathcal{A}P = \mathcal{A}.$$

Thus, $P\mathcal{A} = \mathcal{A}(I - P)$. Now, because $P = \mathcal{C} - P\mathcal{A}$, we may write

$$P = \mathcal{C} - \mathcal{A}(I - P). \tag{4.3}$$

Since $\mathcal{A}$ maps $L^2(b\Omega)$ into $C^\infty(b\Omega)$, we see that $\mathcal{A}(I - P)$ also has this property. Because both $\mathcal{A}(I - P)$ and $\mathcal{C}$ preserve the space $C^\infty(b\Omega)$ we may say the same about the Szegő projection.

Theorem 4.2. *The Szegő projection maps $C^\infty(b\Omega)$ into itself.*

Let $H^2(b\Omega)^\perp$ denote the orthogonal complement of the Hardy space in $L^2(b\Omega)$. Suppose that $v \in H^2(b\Omega)^\perp$. Then it follows that $0 = \langle \mathcal{C}u, v\rangle_b = \langle u, \mathcal{C}^* v\rangle_b$ for all $u \in L^2(b\Omega)$, and therefore, that $\mathcal{C}^* v = 0$. Hence, formula (3.2) for $\mathcal{C}^*$ reveals that $v = \overline{T}\,\overline{H}$ where $H = \mathcal{C}(\overline{Tv})$ is an element of the Hardy space.

Conversely, we showed in the first proof of the Kerzman-Stein formula that any function of the form $v = \overline{T}\,\overline{H}$, where $H \in H^2(b\Omega)$, is orthogonal to the Hardy space. We have almost proved the following theorem.

Theorem 4.3. *A function $u \in L^2(b\Omega)$ has an orthogonal decomposition*

$$u = h + \overline{T}\,\overline{H}$$

where $h = Pu \in H^2(b\Omega)$ and $\overline{T}\,\overline{H} \in H^2(b\Omega)^\perp$. Furthermore $H = P(\bar{u}\overline{T})$. If u is in $C^\infty(b\Omega)$, so are h and H. Functions of the form $\overline{TG}$ where $G \in A^\infty(\Omega)$ form a dense subspace of $H^2(b\Omega)^\perp$.

The only part of Theorem 4.3 that we have not yet proved is that $H = P(\bar{u}\overline{T})$. This can be seen by multiplying the orthogonal decomposition $u = h + \overline{HT}$ by T and then taking the complex conjugate. This gives $\overline{uT} = H + \overline{Th}$, that is an orthogonal decomposition for $\overline{uT}$ in which H appears as the holomorphic part.

Let $P^{\perp} = I - P$ denote the orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)^{\perp}$. Theorem 4.3 reveals that

$$P^{\perp}u = \overline{T}\,\overline{P(\bar{u}\overline{T})}. \tag{4.4}$$

We close this chapter by mentioning that Theorem 4.2 can be localized in the same way that we localized Theorem 3.2 to yield Theorem 3.3. Indeed, since $P = \mathcal{C} - \mathcal{A}(I - P)$, and since $\mathcal{A}$ maps $L^2(b\Omega)$ into $C^{\infty}(b\Omega)$, any local regularity property that $\mathcal{C}$ has is passed on to P. Hence, when the localization argument used to deduce Theorem 3.3 is applied in $L^2(b\Omega)$ to the Cauchy transform, the following theorem is obtained. We will say that a function v in $L^2(b\Omega)$ is C^{∞} smooth on an open arc $\Gamma \subset b\Omega$ if there is a function $\tilde{v}$ on Γ that is C^{∞} smooth such that v is equal almost everywhere on Γ to $\tilde{v}$.

Theorem 4.4. *Suppose that u is a function in $L^2(b\Omega)$ that, on an open arc Γ in $b\Omega$, is C^{∞} smooth. Then $\mathcal{C}u$ and Pu are C^{∞} smooth on Γ.*

5

The Kerzman-Stein operator and kernel

The missing ingredient in the proofs of the theorems in Chapter 4 is the proof that the Kerzman-Stein operator $\mathcal{A}$ is represented by a kernel function $A(z,\zeta)$ that is C^∞ as a function of $(z,\zeta) \in b\Omega \times b\Omega$. To prove this result, we need formula (3.2) which expresses $\mathcal{C}^*$ in terms of $\mathcal{C}$ and we need the classical Plemelj formula which gives an alternate method to that of Chapter 3 for expressing the boundary values of a Cauchy transform.

For a point $z_0 \in b\Omega$ and for small $\epsilon > 0$, let $\gamma_\epsilon(z_0)$ denote the part of the boundary of Ω that is not contained in the disc of radius ϵ about z_0, parameterized in the same sense as $b\Omega$. If $u \in C^\infty(b\Omega)$, we will let $\mathcal{H}u$ denote the function on $b\Omega$ given by the following principal value integral. (The $\mathcal{H}$ stands for "Hilbert." This operator is closely related to the classical Hilbert transform.) If $z_0 \in b\Omega$, then

$$(\mathcal{H}u)(z_0) = \textbf{P.V.}\ \frac{1}{2\pi i}\int_{b\Omega} \frac{u(\zeta)}{\zeta - z_0}\, d\zeta$$

which is defined via

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\gamma_\epsilon(z_0)} \frac{u(\zeta)}{\zeta - z_0}\, d\zeta.$$

We will now prove the Plemelj Theorem, which says that this limit exists and that

$$(\mathcal{C}u)(z_0) = \frac{1}{2}u(z_0) + (\mathcal{H}u)(z_0). \tag{5.1}$$

The Plemelj formula is easy to verify if $u(z)$ is a constant function. Indeed, let C_ϵ denote the curve that traces out the part of the circle of radius ϵ about z_0 that lies inside Ω so that $\gamma_\epsilon(z_0) \cup C_\epsilon$ is a curve that represents the boundary of $\Omega - D_\epsilon(z_0)$ parameterized in the positive sense. The contour integral of $1/(\zeta - z_0)$ about $\gamma_\epsilon \cup C_\epsilon$ is zero and it is easy to check that the part of this integral about C_ϵ tends to $-\pi i$ as ϵ tends to zero. Now that the formula is established for constant functions, we may consider the Cauchy transform of $u - u(z_0)$. Since this function vanishes at z_0, the integrand in the Cauchy transform is not singular at z_0. By letting a point z in Ω approach z_0 along the inward pointing normal to $b\Omega$ at z_0, we see that $[\mathcal{C}(u-u(z_0))](z)$ tends to the integral

$$\frac{1}{2\pi i}\int_{b\Omega}\frac{u(\zeta)-u(z_0)}{\zeta-z_0}\,d\zeta.$$

But we also know that $[\mathcal{C}(u-u(z_0))](z)$ tends to $(\mathcal{C}u)(z_0)-u(z_0)$. Hence,

$$\begin{aligned}(\mathcal{C}u)(z_0) &= u(z_0)+\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\gamma_\epsilon(z_0)}\frac{u(\zeta)-u(z_0)}{\zeta-z_0}\,d\zeta\\ &= u(z_0)+\mathbf{P.V.}\ \frac{1}{2\pi i}\int_{b\Omega}\frac{u}{\zeta-z_0}\,d\zeta-\mathbf{P.V.}\ \frac{1}{2\pi i}\int_{b\Omega}\frac{u(z_0)}{\zeta-z_0}\,d\zeta\\ &= u(z_0)+(\mathcal{H}u)(z_0)-\frac{1}{2}u(z_0)\end{aligned}$$

and the proof of (5.1) is complete.

Let

$$A(z,\zeta)=\frac{1}{2\pi i}\left(\frac{T(\zeta)}{\zeta-z}-\frac{\overline{T(z)}}{\bar\zeta-\bar z}\right)\qquad \text{for } z,\zeta\in b\Omega.$$

If we combine the Plemelj formula with formula (3.2) for $\mathcal{C}^*$, we obtain

$$\begin{aligned}\mathcal{A}u=\mathcal{C}u-\mathcal{C}^*u &= \mathcal{H}u+\overline{T}\,\overline{\mathcal{H}(\bar u\overline{T})}\\ &= \mathbf{P.V.}\ \frac{1}{2\pi i}\int_{\zeta\in b\Omega}A(z,\zeta)u(\zeta)\,ds.\end{aligned}$$

We will now prove that $A(z,\zeta)$ is infinitely differentiable as a function of (z,ζ). It then will follow that the principal value integral for $\mathcal{A}u$ above is a standard integral and the proof will be finished. To simplify the computations, let us suppose that the boundary curves of Ω have been parameterized with respect to *arc length.* We wish to see that $A(z(t),z(s))$ is C^∞ smooth as a function of (t,s) as t and s range over the various parameter intervals. We should also check that the values of these functions and their derivatives agree at the endpoints of the parameter intervals. However, since nothing special is really happening at the endpoints of the parameter intervals, we may assume that t and s stay in the interior of the intervals. Since $A(z(t),z(s))$ is clearly C^∞ smooth when t and s are in different parameter intervals, and when t and s belong to the same interval and $t\neq s$, we need only show that $A(z(t),z(s))$ is C^∞ in (t,s) when t and s belong to the interior of the same parameter interval and $|t-s|<\epsilon$ for some small $\epsilon>0$.

The proof rests on the elementary fact that, if $Y(s,t)$ is C^∞ in (s,t) and $Y(t,t)=0$, then $Y(s,t)=(s-t)W(s,t)$ where $W(s,t)$ is also C^∞ in (s,t). To understand this fact, write

$$Y(s,t)=\int_t^s \frac{\partial Y}{\partial x_1}(x_1,t)\,dx_1$$

and make the change of variables $x_1=t+u(s-t)$ to get

$$Y(s,t)=\left(\int_0^1 \frac{\partial Y}{\partial x_1}(t+u(s-t),t)\,du\right)(s-t).$$

The function represented by the integral is easily seen to be C^∞ smooth in both variables s and t.

Since $z(s)-z(t)$ is a C^∞ function of (s,t) that vanishes at (t,t), we may use the fact above to see that the difference quotient $Q(s,t)=(z(s)-z(t))/(s-t)$ is a C^∞ function of (s,t). Note that $Q(t,t)=z'(t)$, and therefore, that $Q(s,t)$ is nonvanishing when t and s are close together. Also, since s represents arc length, it follows that $T(z(s))=z'(s)$. We may now write

$$2\pi i\, A(z(t),z(s))=\frac{1}{s-t}\left(\frac{z'(s)}{Q(s,t)}-\frac{\overline{z'(t)}}{\overline{Q(s,t)}}\right).$$

Thus, we see that $2\pi i(s-t)A(z(t),z(s))$ is equal to a C^∞ function $R(s,t)$. Furthermore, $R(t,t)=1-1=0$; so $R(s,t)=(s-t)X(s,t)$ where $X(s,t)$ is also C^∞ smooth. After dividing out by $(s-t)$, we conclude that $2\pi i\, A(z(t),z(s))=X(s,t)$ is C^∞ in (s,t), and this is precisely what it means to say that $A(z,\zeta)$ is C^∞ smooth on $b\Omega\times b\Omega$. The proof is complete.

Notice that $A(z,w) = -\overline{A(w,z)}$. It follows that $A(z,z)$ is a pure imaginary number. In fact, by studying the Taylor expansion of $A(z(t),z(s))$ in the t variable at $t=s$, it is not hard to show that $A(z,z)=0$. Since we will not need this fact, we will not prove it.

It is worth pointing out that in the course of our work above, we have proved the following classical theorem.

Theorem 5.1. *The transform $\mathcal{H}$ maps $C^\infty(b\Omega)$ into itself and extends to be a bounded operator on $L^2(b\Omega)$.*

6

The classical definition of the Hardy space

We have defined the Hardy space as a subspace of $L^2(b\Omega)$. We will now identify the Hardy space with a space of holomorphic functions on Ω. If $u \in L^2(b\Omega)$, then $\mathcal{C}u$ has been defined to be the limit in $L^2(b\Omega)$ of $\mathcal{C}u_j$ where u_j is any sequence of functions in $C^\infty(b\Omega)$ converging to u in $L^2(b\Omega)$. The functions $\mathcal{C}u_j$ are in $A^\infty(\Omega)$ and it is easy to see that they converge uniformly on compact subsets of Ω to a holomorphic function H. Although we have been thinking of the Cauchy transform $\mathcal{C}$ as an operator on $L^2(b\Omega)$, let us agree to abuse our notation and also use the symbol $\mathcal{C}$ to represent the classical Cauchy integral,

$$(\mathcal{C}u)(z) = \frac{1}{2\pi i}\int_{\zeta\in b\Omega} \frac{u(\zeta)}{\zeta - z}\,d\zeta$$

for $z \in \Omega$. The holomorphic function H is given by $H(z) = (\mathcal{C}u)(z)$. The purpose of this chapter is to show that this dual use of the symbol $\mathcal{C}$ is not an abuse. In fact, we will show that H has L^2 boundary values given by $\mathcal{C}u$. Furthermore, there is a one-to-one correspondence between elements h of $H^2(b\Omega)$ and holomorphic functions H on Ω arising as their Cauchy integrals. In this chapter, we will use lowercase letters to denote functions on the boundary in $L^2(b\Omega)$ and we will let uppercase letters denote holomorphic functions on Ω. In particular, if h and H are used in the same paragraph, they will be related via the Cauchy integral formula $H(z) = (\mathcal{C}h)(z)$. When this chapter is finished, we will be justified to use the same symbol for h and H.

For $\epsilon > 0$, let Ω_ϵ denote the set of points in Ω that are a distance of more than ϵ from the boundary of Ω. For small $\epsilon > 0$, Ω_ϵ is a bounded domain with C^∞ smooth boundary. Let $z(t)$ denote a parameterization of the boundary of Ω in the standard sense. (We are dropping the subscript j's to streamline the notation. It is understood that $z(t)$ represents parameterizations of all the boundary components of Ω.) The function $z_\epsilon(t) = z(t) + i\epsilon T(z(t))$ parameterizes the curve obtained by allowing a point at a distance ϵ along the inward pointing normal to $z(t) \in b\Omega$ to trace out a curve as $z(t)$ ranges over the boundary. It is a standard fact about curves in the plane that, if ϵ is sufficiently small, $z_\epsilon(t)$ parameterizes the boundary of Ω_ϵ in the standard sense. Let $\delta > 0$ be a small positive number such that Ω_ϵ is a C^∞ domain parameterized by $z_\epsilon(t)$ when $0 < \epsilon < \delta$.

Classically, the Hardy space was defined to be the space of holomorphic functions H on Ω such that

$$\sup_{0<\epsilon<\delta} \left(\int |H(z_\epsilon(t))|^2 |z_\epsilon'(t)| \, dt \right)^{1/2}$$

is finite. We will prove that the space of holomorphic functions on Ω satisfying this *classical Hardy condition* is equal to the space of functions given by Cauchy integrals of functions in the Hardy space as we have defined it.

Theorem 6.1. *Functions that satisfy the classical Hardy condition are Cauchy integrals of functions in $H^2(b\Omega)$. In particular, bounded holomorphic functions are Cauchy integrals of functions in $H^2(b\Omega)$.*

Proof. Suppose H is a holomorphic function that satisfies the classical Hardy condition. The family of functions $\{u_\epsilon\}_{\epsilon>0}$ on $b\Omega$ defined via $u_\epsilon(z(t)) = H(z_\epsilon(t))$ is easily seen to be bounded in $L^2(b\Omega)$ because $|z'(t)| < c|z_\epsilon'(t)|$ with c independent of ϵ. We may therefore find a sequence ϵ_j tending to zero such that u_{ϵ_j} converges weakly to a function u in $L^2(b\Omega)$. We now claim that $H = \mathcal{C}u$. To see this, let $\mathcal{C}_\epsilon$ denote the Cauchy transform associated to the domain Ω_ϵ. If $a \in \Omega$ and ϵ is small, we may write

$$(\mathcal{C}u)(a) - H(a) = (\mathcal{C}u - \mathcal{C}_\epsilon H)(a) = \mathcal{C}(u - u_\epsilon)(a) + (\mathcal{C}u_\epsilon - \mathcal{C}_\epsilon H)(a).$$

Now, if we let $\epsilon \to 0$ through values yielding terms in our weakly convergent subsequence, it is clear that $\mathcal{C}(u - u_\epsilon)(a)$ tends to zero. Next, we may write out the integral defining $(\mathcal{C}u_\epsilon - \mathcal{C}_\epsilon H)(a)$ in terms of t to obtain

$$\frac{1}{2\pi i} \int H(z_\epsilon(t)) \left(\frac{z'(t)}{z(t) - a} - \frac{z_\epsilon'(t)}{z_\epsilon(t) - a} \right) \, dt.$$

This quantity tends to zero because $H(z_\epsilon(t))$ is bounded in L^2 (as a function of t) and the other term in the integral tends uniformly to zero as ϵ tends to zero. Hence $H = \mathcal{C}u$. We have shown that H is the Cauchy integral of a function in $L^2(b\Omega)$. We will finish the proof by showing that $u \in H^2(b\Omega)$, i.e., that $\mathcal{C}u = u$. To see this, we will prove that u is orthogonal to $H^2(b\Omega)^\perp$. In fact, it will be enough to show that $\langle u, \overline{TG} \rangle_{b\Omega} = 0$ for all $G \in A^\infty(\Omega)$ since, by Theorem 4.3, functions of this form are dense in $H^2(b\Omega)^\perp$. Let T_ϵ denote the unit tangent vector function associated to $b\Omega_\epsilon$. By writing out the integrals and using the weak convergence of u_ϵ to u as we did above, we will see that $\langle u, \overline{TG}\rangle_{b\Omega} - \langle H, \overline{T_\epsilon G}\rangle_{b\Omega_\epsilon}$ tends to zero as ϵ tends to zero through values indexing the weakly convergent subsequence. But, $\langle H, \overline{T_\epsilon G}\rangle_{b\Omega_\epsilon} = 0$ by Cauchy's theorem. Hence, it will follow that $u \in H^2(b\Omega)$, and the proof will be finished. Now,

$$\langle u, \overline{TG}\rangle_{b\Omega} - \langle H, \overline{T_\epsilon G}\rangle_{b\Omega_\epsilon} = \langle u - u_\epsilon, \overline{TG}\rangle_{b\Omega} + \mathcal{I}$$

where

$$\mathcal{I} = \int H(z_\epsilon(t))\,(z'(t)G(z(t)) - z'_\epsilon(t)G(z_\epsilon(t)))\ dt.$$

The term $\langle u - u_\epsilon, \overline{TG}\rangle_{b\Omega}$ tends to zero as ϵ tends to zero because of the weak convergence of u_ϵ to u, and the term $\mathcal{I}$ tends to zero because $H(z_\epsilon(t))$ is bounded in L^2 (with respect to dt) and the other function in the integrand tends uniformly to zero as ϵ tends to zero. The proof is finished. $\square$

The function u in the proof above can be thought of as the $L^2(b\Omega)$ boundary values of H. We will make this association more clear in a moment. First, however, to round out the picture, we must show that a function that is the Cauchy integral of a function in $H^2(b\Omega)$ satisfies the classical Hardy condition.

Theorem 6.2. *Cauchy integrals of functions in $H^2(b\Omega)$ are holomorphic functions on Ω that satisfy the classical Hardy condition.*

Proof. Suppose that $H \in A^\infty(\Omega)$. Define

$$N(\epsilon) = \int |H(z_\epsilon(t))|^2 |z'_\epsilon(t)|\, dt.$$

Notice that $N(0) = \|H\|^2$. We want to show that $N(\epsilon)$ satisfies an estimate of the form $N(\epsilon) \le CN(0)$ where $C > 0$ is a constant that does not depend on H. Having established this inequality for functions in $A^\infty(\Omega)$, we can approximate a function $g \in H^2(b\Omega)$ by functions $G_j \in A^\infty(\Omega)$. Letting G denote the Cauchy integral of g, it then follows from the $L^2(b\Omega)$ convergence of G_j to g on $b\Omega$ that the G_j converge uniformly on compact subsets of Ω to G. Hence, for a fixed ϵ, the G_j converge to G in the L^2 norm on $b\Omega_\epsilon$. Now, by writing the uniform estimate applied to the G_j and letting $j \to \infty$, we see that the $L^2(b\Omega_\epsilon)$ norm of G is bounded by the same constant C times the $H^2(b\Omega)$ norm of g.

We now return to proving the estimate $N(\epsilon) \le CN(0)$ when $H \in A^\infty(\Omega)$. Note that $N(\epsilon)$ is a C^∞ function of ϵ on $0 \le \epsilon < \delta$ and that $N(\epsilon) \ge 0$ if $\epsilon > 0$. We now wish to compute the derivative $N'(\epsilon)$. In order to do this, we will need to know an elementary property of the curves that bound Ω_ϵ. It is that, at a point $z_\epsilon(t) = z(t) + i\epsilon T(z(t))$, the inward pointing unit normal vector to $b\Omega_\epsilon$ is the same as the inward pointing unit normal vector to $b\Omega$ at $z(t)$. From this, it follows that $T_\epsilon(z_\epsilon(t)) = T(z(t))$. Let D_ϵ denote the differentiation with respect to ϵ operator. Note that

$$D_\epsilon\,(z_\epsilon(t)) = iT(z(t)) = iT_\epsilon(z_\epsilon(t)). \tag{6.1}$$

Now, by differentiating under the integral, we obtain

$$\begin{aligned} N'(\epsilon) = &\int H'(z_\epsilon(t))\,\overline{H(z_\epsilon(t))}D_\epsilon\,(z_\epsilon(t))\;\;|z_\epsilon'(t)|\,dt \\ &+ \int H(z_\epsilon(t))\,\overline{H'(z_\epsilon(t))D_\epsilon\,(z_\epsilon(t))}\;|z_\epsilon'(t)|\,dt \\ &+ \int |H(z_\epsilon(t))|^2 D_\epsilon\,(|z_\epsilon'(t)|)\;dt. \end{aligned}$$

Using formula (6.1) and the complex Green's formula, the first of these three integrals is seen to be equal to

$$i\int_{b\Omega_\epsilon} H'\,\bar{H}\,dz = i\iint_{\Omega_\epsilon} H'\,\overline{H'}\,d\bar{z}\wedge dz,$$

which is less than or equal to zero because $i\,d\bar{z}\wedge dz = -2\,dx\wedge dy$. The second of the three integrals is just the complex conjugate of the first. Since the first integral is a negative real number, it follows that so is the second integral. Hence, $N'(\epsilon)$ is seen to be less than or equal to the third integral. Let M denote an upper bound for the quantity

$$\frac{D_\epsilon|z_\epsilon'(t)|}{|z_\epsilon'(t)|}$$

as ϵ ranges over $0 \le \epsilon < \delta$ and t ranges over its domain. We may now estimate the third integral to obtain

$$N'(\epsilon) \le MN(\epsilon) \tag{6.2}$$

for $0 \le \epsilon < \delta$.

The next argument should be familiar to anyone who has been in the same room with an ODE. Multiplying inequality (6.2) by $e^{-M\epsilon}$ and subtracting, we obtain

$$e^{-M\epsilon}N'(\epsilon) - Me^{-M\epsilon}N(\epsilon) \le 0.$$

Integrating this inequality between 0 and ϵ yields

$$e^{-M\epsilon}N(\epsilon) - N(0) \le 0,$$

and the proof of the desired inequality is complete, and hence, so is the proof of the theorem. □

Although we will not need to know it, the following fact is interesting. It is not too hard to show that, if $z(t)$ is the parameterization of $b\Omega$ with respect to arc length, then

$$D_\epsilon\left(|z'_\epsilon(t)|\right) = i\frac{z''(t)}{z'(t)}.$$

(Some of the tricks used in this computation include

$$|z'_\epsilon(t)| = \frac{|z'(t)|}{z'(t)}\, z'_\epsilon(t)$$

and $|z'(t)| \equiv 1$.) Incorporating this identity in the expression above for N' yields that

$$N'(\epsilon) = -4\iint_{\Omega_\epsilon} |H'|^2\, dx \wedge dy + i\int |H(z_\epsilon(t))|^2 \frac{z''(t)}{z'(t)}\, dt.$$

It is interesting that, if Ω is the unit disc, then $z''(t)/z'(t) = i$, and therefore, N is a strictly decreasing function of ϵ.

To really understand the Hardy space, we need to prove one more theorem. This theorem will allow us to say, in a strong sense, that $H^2(b\Omega)$ is equal to the space of holomorphic functions on Ω with L^2 boundary values.

Given $h \in H^2(b\Omega)$, let $H(z) = (\mathcal{C}h)(z)$ be the holomorphic function on Ω given by the Cauchy integral of h and let $\{u_\epsilon\}$ denote the family of functions on $b\Omega$ defined via $u_\epsilon(z(t)) = H(z_\epsilon(t))$ as above.

Theorem 6.3. *If $h \in H^2(b\Omega)$, then $u_\epsilon \to h$ in $L^2(b\Omega)$ as $\epsilon \to 0$.*

Proof. We have shown that there is a constant C (which is independent of h and ϵ) such that $\|u_\epsilon\| \le C\|h\|$. Let $\lambda > 0$ and let $G \in A^\infty(\Omega)$ be such that $\|h - G\| < \lambda$. Let U_ϵ denote the family of functions defined on $b\Omega$ corresponding to G, i.e., $U_\epsilon(z(t)) = G(z_\epsilon(t))$. Now,

$$\|u_\epsilon - h\| \le \|u_\epsilon - U_\epsilon\| + \|U_\epsilon - G\| + \|G - h\|.$$

The first term is less than or equal to $C\lambda$, the second term tends to zero as $\epsilon \to 0$ because G is continuous on $\overline{\Omega}$, and the third term is less than λ. The proof is complete. □

The association of a classical Hardy function H to its L^2 boundary values sets up a one-to-one correspondence between the space of classical

Hardy functions and $H^2(b\Omega)$. Hence, we are justified in using the same symbol h to denote a function in $H^2(b\Omega)$ and the holomorphic function on Ω that is its Cauchy integral.

We will need the following fact later, which is a direct consequence of the work above. (Recall that if a sequence converges in L^2, then there is a subsequence converging pointwise almost everywhere.)

Theorem 6.4. *If h is a holomorphic function on Ω such that $|h| < 1$ on Ω, then $|h| \leq 1$ on $b\Omega$ as a function in $H^2(b\Omega)$.*

This is also an opportune moment to state another theorem that we will need later when we study the Bergman space in Chapter 15. Let A_δ denote the annular region consisting of points in Ω that are within a distance of δ to the boundary of Ω. Using the notation set up in the proof of Theorem 6.2, notice that

$$\int_0^\delta N(\epsilon)\, d\epsilon = \int_0^\delta \left(\int |H(z_\epsilon(t))|^2 |z'_\epsilon(t)|\, dt \right) d\epsilon = \iint_{A_\delta} |H|^2 \, dx\, dy$$

if $H \in A^\infty(\Omega)$. Hence, the estimate $N(\epsilon) \leq C\, N(0)$ derived in the course of the proof of Theorem 6.2 reveals that there is a constant C such that the L^2 norm of H with respect to Lebesgue area measure on A_δ is bounded by a constant times the $H^2(b\Omega)$ norm of H. Since the value of $H(z)$ at a point z in Ω that is at a distance greater than δ from the boundary is easily bounded by a uniform constant times the $H^2(b\Omega)$ norm of H via the Cauchy integral formula, we have proved the following theorem.

Theorem 6.5. *If h is in $H^2(b\Omega)$, then $|h|^2$ is integrable over Ω with respect to Lebesgue area measure, and there is a constant C which does not depend on h such that*

$$\iint_\Omega |h|^2 \; dx\, dy \leq C \|h\|^2.$$

7

The Szegő kernel function

For $a \in \Omega$ and $z \in b\Omega$, let $C_a(z)$ denote the *complex conjugate of*

$$\frac{1}{2\pi i}\frac{T(z)}{z-a}.$$

With this notation, C_a is the kernel that defines the Cauchy integral in the sense that $(\mathcal{C}u)(a) = \langle u, C_a\rangle_b$. If h is a function in the Hardy space, we have identified h with the holomorphic function on Ω given by $\mathcal{C}h$. The value of h at a point $a \in \Omega$ can be computed via

$$h(a) = (\mathcal{C}h)(a) = \langle h, C_a\rangle_b = \langle h, P(C_a)\rangle_b = \langle h, S_a\rangle_b$$

where $S_a = PC_a$. The function $S(z,a)$ defined to be $S(z,a) = S_a(z)$ is called the *Szegő kernel function*.

Another way to think of the Szegő kernel is via the Riesz representation theorem. The formula $h(a) = \langle h, C_a\rangle_b$ shows that $|h(a)| \le \|h\|\|C_a\|$, and therefore, that evaluation at $a \in \Omega$ is a continuous linear functional on $H^2(b\Omega)$. Hence, there is a unique function $E_a \in H^2(b\Omega)$ that represents this functional in the sense that $h(a) = \langle h, E_a\rangle_b$ for all $h \in H^2(b\Omega)$. We showed above that PC_a also has this property. Because E_a is uniquely determined, it follows that the function E_a defined by the representing property is the same as the function S_a we defined above as PC_a.

Because P maps $C^\infty(b\Omega)$ into itself, it follows that $S_a(z)$ is in $A^\infty(\Omega)$ as a function of z when $a \in \Omega$ is fixed. We will now show that $S(z,a)$ is continuous as a function of *both variables* (z,a) on $\Omega\times\Omega$ and that $S(z,a)$ is antiholomorphic in a. Notice that if $a, z \in \Omega$, then $S_a(z) = \langle PC_a, C_z\rangle_b$, and, using the identity

$$S_{a_1}(z) - S_{a_2}(z) = \langle P(C_{a_1} - C_{a_2}), C_z\rangle_b$$

and the estimate $\|Pu\| \le \|u\|$, we may therefore estimate

$$|S_{a_1}(z) - S_{a_2}(z)| \le \|C_{a_1} - C_{a_2}\|\|C_z\|.$$

If a_1 and a_2 are restricted to be in a compact subset of Ω, then $\|C_{a_1} -$

$C_{a_2}\|$ can be made uniformly small by demanding that $|a_1 - a_2|$ be small. Furthermore, $\|C_z\|$ is uniformly bounded when z is restricted to be in a compact set. Using these facts and the basic estimates, it is easy to show that $|S(z_1, a_1) - S(z_2, a_2)|$ is small when (z_1, a_1) is close to (z_2, a_2) in a compact subset of $\Omega \times \Omega$. Hence $S(z, a)$ is continuous on $\Omega \times \Omega$. To see that $S(z, a)$ is antiholomorphic in a, we compute a difference quotient using the ideas above to obtain

$$\frac{S_a(z) - S_{a_0}(z)}{\bar{a} - \bar{a}_0} = \langle P\left(\frac{C_a - C_{a_0}}{\bar{a} - \bar{a}_0}\right), C_z\rangle_b.$$

Since $(\bar{a} - \bar{a}_0)^{-1}(C_a - C_{a_0})$ tends in $L^2(b\Omega)$ to $C'(a_0)$ where $[C'(a)](\zeta)$ is defined to be equal to

$$\frac{\partial}{\partial \bar{a}} C_a(\zeta) = -\frac{1}{2\pi i} \frac{\overline{T(\zeta)}}{(\bar{\zeta} - \bar{a})^2},$$

it follows that the difference quotient tends to $\langle P(C'(a_0)), C_z\rangle_b$, and this shows that $S(z, a)$ is antiholomorphic in a. By repeating this argument, it can be shown that $S(z, a)$ is in fact C^∞ smooth as a function of both variables (z, a) on $\Omega \times \Omega$. The proof boils down to showing that it is permissible to differentiate under the operators and inner product in the expression $S_a(z) = \langle PC_a, C_z\rangle_b$, and this follows from the L^2 estimate for P and the uniform differentiability of the Cauchy kernel. Later in the book (Chapter 26), we will prove a much stronger result. We will prove that $S(z, a)$ is in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \mathcal{D})$ where $\mathcal{D} = \{(z, z) : z \in b\Omega\}$ is the boundary diagonal.

It is not hard to see that the Szegő kernel is hermitian symmetric, i.e., that $S(a, b) = \overline{S(b, a)}$ for all $a, b \in \Omega$. Indeed, using the representing property of S_b, we obtain

$$S_a(b) = \langle S_a, S_b\rangle_b,$$

and this is clearly the complex conjugate of

$$S_b(a) = \langle S_b, S_a\rangle_b,$$

which follows from the representing property of S_a. Note that, if we set $a = b$ in the work above, we obtain

$$S(a, a) = S_a(a) = \int_{b\Omega} |S_a|^2 \, ds.$$

This identity shows that $S(a, a) > 0$. Indeed, since the L^2 pairing of S_a with the function that is identically one gives the value one, it follows that S_a cannot be identically zero as a function in $L^2(b\Omega)$. Hence,

$S(a,a)>0$ for each $a\in\Omega$. We will need this fact later when we study the relationship between the Szegő kernel and the Riemann mapping function.

For $h\in H^2(b\Omega)$, using the hermitian symmetry of $S(z,a)$, we may write

$$h(a)=\int_{z\in b\Omega} S(a,z)h(z)\,ds.$$

The Szegő kernel is the kernel for the Szegő projection in the classical sense of integral operators because

$$(Pu)(a)=\langle Pu,S_a\rangle_b=\langle u,S_a\rangle_b=\int_{z\in b\Omega} S(a,z)u(z)\,ds.$$

Theorem 4.3 allows us to define another important kernel function. Consider the orthogonal decomposition of the Cauchy kernel $C_a(\zeta)$. We know that $(PC_a)(\zeta)=S(\zeta,a)$. Therefore, the orthogonal decomposition for C_a given by Theorem 4.3 is $C_a=S_a+\overline{H_aT}$, where H_a is in $A^\infty(\Omega)$. Conjugating the orthogonal decomposition of C_a and using the hermitian symmetry of $S(z,a)$ yields

$$\frac{1}{2\pi i}\frac{T(\zeta)}{\zeta-a}=S(a,\zeta)+H_a(\zeta)T(\zeta).$$

The function $H_a(\zeta)$ is in $A^\infty(\Omega)$ as a function of ζ for fixed $a\in\Omega$. Also, it can be seen from the decomposition formula that H_a is holomorphic in $a\in\Omega$ for fixed $\zeta\in b\Omega$ because the same is true of the other terms in the formula. *Garabedian's L kernel* is defined via

$$\frac{1}{i}L(\zeta,a)=\frac{1}{2\pi i}\frac{1}{\zeta-a}-H_a(\zeta).$$

When the function H_a is expressed in terms of the Garabedian kernel and when this formula is plugged back into the orthogonal decomposition of the Cauchy kernel, the following identity is obtained.

The Szegő and Garabedian kernels are related via

$$S(a,\zeta)=\frac{1}{i}L(\zeta,a)T(\zeta)\qquad \text{for } a\in\Omega,\ \zeta\in b\Omega. \tag{7.1}$$

This identity is very important and is at the heart of many of the applications of the Szegő kernel to problems in conformal mapping. We will use (7.1) in various forms many times in later chapters and so we take this opportunity to list several different ways to write this identity. Since $|T(\zeta)|=1$, it follows that $1/T(\zeta)=\overline{T(\zeta)}$. Using this fact and the shorthand notation $S_a(\zeta)=S(\zeta,a)$ and $L_a(\zeta)=L(\zeta,a)$, (7.1) can be written in any of the following equivalent ways. Thinking of a as being fixed in Ω and S_a and L_a as functions of $\zeta \in b\Omega$, we have

$$\begin{aligned}
\overline{S_a} &= -iL_aT, \\
S_a &= i\overline{L_aT}, \\
L_a &= i\overline{S_aT}, \\
\overline{L_a} &= -iS_aT.
\end{aligned}$$

Notice that when we write identity (7.1) in the form $L(\zeta, a) = i\,\overline{S_a(\zeta)T(\zeta)}$, Theorem 4.3 shows that $L(\zeta, a)$ is orthogonal to $H^2(b\Omega)$ as a function of ζ. Thus,

$$L(\zeta, a) = P^{\perp}(L(\cdot, a))(\zeta).$$

But $L(\zeta, a) = \frac{1}{2\pi}(\zeta - a)^{-1} - H_a(\zeta)$ where $H_a \in A^{\infty}(\Omega)$, and since $P^{\perp}H_a = 0$, it follows that

$$L(\zeta, a) = (P^{\perp}G_a)(\zeta)$$

where G_a is defined to be

$$G_a(\zeta) = \frac{1}{2\pi(\zeta - a)}.$$

It can be read off from the formula defining the Garabedian kernel that, for fixed $a \in \Omega$, $L(\zeta, a)$ is a meromorphic function of ζ on Ω with a single simple pole at $\zeta = a$ having residue $1/(2\pi)$. Also, $L(\zeta, a)$ extends C^{∞} smoothly up to the boundary as a function of ζ. Because $H_a(\zeta)$ is holomorphic in a, it follows that, for fixed ζ, $L(\zeta, a)$ is holomorphic in a on $\Omega - \{\zeta\}$.

Alternatively, $L(\zeta, a)$ can be seen to be holomorphic in a by differentiating under the operator in the identity $L(\zeta, a) = (P^{\perp}G_a)(\zeta)$ and by using reasoning analogous to that which we used to study the smoothness properties of the Szegő kernel above.

We now show that identity (7.1) characterizes the Szegő and Garabedian kernels.

Theorem 7.1. *Suppose that $\sigma(z)$ is a holomorphic function on Ω that extends continuously to $\overline{\Omega}$ and suppose $\lambda(z)$ is holomorphic on $\Omega - \{a\}$, extends continuously to $b\Omega$, and has a simple pole with residue $1/(2\pi)$ at $z = a$. If*

$$\overline{\sigma(z)} = \frac{1}{i}\lambda(z)T(z) \qquad \text{for } z \in b\Omega, \tag{7.2}$$

then $\sigma(z) = S(z, a)$ and $\lambda(z) = L(z, a)$.