Proof. We may write $\lambda(z) = \frac{1}{2\pi}(z-a)^{-1} + h(z)$ where h is a holomorphic function on Ω that extends continuously to $\overline{\Omega}$. But $\lambda = i\bar{\sigma}\bar{T}$, and therefore λ is orthogonal to holomorphic functions. Thus $\lambda(z) = (P^{\perp}\lambda)(z) = (P^{\perp}(G_a + h))(z) = (P^{\perp}G_a)(z) = L(z,a)$, and (7.1) yields that $\sigma(z) = S(z,a)$. The proof is finished. □

Another way to think of Garabedian's kernel is as the kernel for the orthogonal projection $P^{\perp}$. Indeed, $P^{\perp}u = \overline{HT}$ where $H = P(\overline{T}\bar{u})$. We may evaluate H at a point $a \in \Omega$ by using identity (7.1) in the following computation:

$$\begin{aligned} H(a) = P(\overline{T}\bar{u})(a) &= \int_{\zeta\in b\Omega} S(a,\zeta)\overline{T(\zeta)u(\zeta)}\, ds \\ &= \frac{1}{i}\int_{\zeta\in b\Omega} L(\zeta,a)\overline{u(\zeta)}\, ds. \end{aligned}$$

It is possible to manipulate the identities of this chapter to deduce that $L(a,b) = -L(b,a)$ for $a \neq b$ in Ω. Indeed, $L(a,b)$ is equal to the residue of $2\pi L(z,a)L(z,b)$ at the point a because $L(z,a)$ has a simple pole at a with residue $(2\pi)^{-1}$. The same reasoning shows that $L(b,a)$ is equal to the residue of $2\pi L(z,a)L(z,b)$ at the point b. Hence, we may use the residue theorem to compute,

$$\begin{aligned} L(a,b) + L(b,a) = 2\pi \sum \text{Res } L(z,a)L(z,b) &= \frac{1}{i}\int_{b\Omega} L(z,a)L(z,b)\, dz \\ = \int_{b\Omega} L(z,a)\frac{1}{i}L(z,b)T(z)\, ds &= \int_{b\Omega} L(z,a)S(b,z)\, ds = 0 \end{aligned}$$

because, as mentioned above, $L(z,a)$ is orthogonal to $H^2(b\Omega)$.

All the facts and formulas we have derived for a general domain become particularly simple when the domain is the unit disc.

Theorem 7.2. *The Szegő kernel of the unit disc U is given by*

$$S(z,a) = \frac{1}{2\pi(1-\bar{a}z)}$$

and the Garabedian kernel is given by

$$L(z,a) = \frac{1}{2\pi(z-a)}.$$

A function u in $L^2(bU)$ has an orthogonal decomposition

$$u = h + \overline{H}$$

where $h = Pu$ and $H = z\,P(\bar{z}\bar{u})$. Notice that H is holomorphic on U and vanishes at the origin.

Proof. On the unit disc, the complex unit tangent at a point z in the boundary is given by $T(z) = iz$. Hence the orthogonal decomposition follows immediately from the formulas in Theorem 4.3.

The Szegő kernel is equal to the Szegő projection of the Cauchy kernel. On the unit disc, the Cauchy kernel turns out to be a holomorphic function. Indeed, we may write

$$C_a(z) = -\frac{1}{2\pi i}\frac{\overline{T(z)}}{\bar{z}-\bar{a}} = \frac{1}{2\pi}\frac{\bar{z}}{\bar{z}-\bar{a}}.$$

Now, using the fact that $\bar{z} = 1/z$ when $|z| = 1$, we obtain

$$C_a(z) = \frac{1}{2\pi}\frac{1}{1-\bar{a}z}$$

which is a holomorphic function in $A^\infty(U)$. Hence, it follows that $S_a = PC_a = C_a$ and the formula for the Szegő kernel is proved.

Since $S_a = C_a$, the function H_a in the orthogonal expansion of C_a must be zero. Since

$$L(z,a) = \frac{1}{2\pi}\frac{1}{z-a} - iH_a(z),$$

the formula for the Garabedian kernel is proved. □

The equality $S_a = C_a$ that we just saw is valid on the unit disc is very special. In fact, Kerzman and Stein [K-S] proved that this identity holds for the Szegő kernel of a domain Ω if and only if Ω is equal to a disc.

There are some interesting theorems lurking in the background of Theorem 7.2. For example, if $u \in C^\infty(bU)$, the orthogonal decomposition $u = h + \overline{H}$ gives a harmonic extension of u to the closed disc and this extension must agree with the one given by the classical Poisson integral formula (see Ahlfors [Ah, p. 166]). Since P maps $C^\infty(bU)$ into $A^\infty(U)$, Theorem 7.2 reveals that the harmonic extension of u is in $C^\infty(\overline{U})$ whenever $u \in C^\infty(bU)$ and it is given by $Pu + \bar{z}\,\overline{P(\bar{z}\bar{u})}$. We will return to the problem of finding the harmonic extension of a function defined on the boundary of a more general domain in Chapter 10 where we will relate the solution of this problem to the Szegő projections of simple functions.

8

The Riemann mapping function

In this chapter, we will assume that Ω is a bounded *simply connected* domain in $\mathbb{C}$ with C^∞ smooth boundary. We wish to illustrate as quickly and easily as possible that the Szegő and Garabedian kernels are intimately tied to questions in conformal mapping.

The Riemann mapping theorem asserts that, for a point $a \in \Omega$, there is a one-to-one holomorphic mapping f of Ω onto $D_1(0)$ such that $f(a) = 0$. If we require that $f'(a)$ be real and positive, then f is uniquely determined. We will call this mapping *the* Riemann mapping function associated to a.

In order to make the proof of the next theorem as short and easy as possible, we will assume Carathéodory's theorem, which states that Riemann maps of the types of domains we are studying extend continuously to the boundary. Later we will prove the Riemann mapping theorem and the theorem below from first principles without relying on Carathéodory's result.

Theorem 8.1. *The Riemann mapping function f associated to a point a in a bounded simply connected domain Ω with C^∞ smooth boundary is given by*

$$f(z) = \frac{S(z,a)}{L(z,a)}$$

where $S(z,a)$ is the Szegő kernel and $L(z,a)$ is the Garabedian kernel associated to Ω.

Proof. Since $S(a,a)$ is a positive real number, the function

$$\lambda(z) = S(z,a)/f(z)$$

is a meromorphic function on Ω which extends continuously to the boundary and which has a single simple pole at the point a. Furthermore, the function $\sigma(z) = f(z)L(z,a)$ is holomorphic on Ω and extends continuously to the boundary. Since f maps the boundary of Ω into the unit circle, it follows that $1/f = \bar{f}$ on the boundary of Ω, and since $S(z,a)$ and $L(z,a)$ satisfy formula (7.1), we see that σ and λ satisfy identity (7.2). Thus Theorem 7.1 implies that $\lambda(z) = cL(z,a)$ for some constant c, i.e., that $cf(z) = S(z,a)/L(z,a)$. Now, because formula (7.1) implies that $|S(z,a)| = |L(z,a)|$ for $z \in b\Omega$, we conclude that $|c| = 1$. Finally, because $f'(a)$ and $S(a,a)$ are real and positive, and because the residue of $L(z,a)$ at a is $1/(2\pi)$, it follows that c is real and positive. Thus, $c = 1$, and the proof is complete. □

We have assumed Carathéodory's theorem about continuous extension of the Riemann map in order to prove Theorem 8.1. Soon, we will prove from first principles a much stronger result than Carathéodory's theorem about the boundary behavior of the Riemann map. We will prove the following theorem of Painlevé. (It is interesting to note that Painlevé proved his theorem before Carathéodory proved his. See [Be-K] for the history of this question.)

Theorem 8.2. *The Riemann mapping function f, mapping a bounded simply connected domain Ω with C^∞ smooth boundary onto the unit disc, is C^∞ smooth up to the boundary. Furthermore, f' is nonvanishing on $\overline{\Omega}$. Hence, f^{-1} is C^∞ smooth up to the boundary of the unit disc.*

9

A density lemma and consequences

We now go back to studying the Szegő kernel function on a general bounded domain Ω with C^∞ smooth boundary. In particular, Ω is again allowed to be multiply connected.

For fixed $a \in \Omega$, we will let $S_a(z)$ denote the function of z given by $S_a(z) = S(z,a)$. Let Σ denote the (complex) linear span of the set of functions $\{S_a(z) : a \in \Omega\}$. It is easy to see that Σ is a dense subspace of $H^2(b\Omega)$. Indeed, if $h \in H^2(b\Omega)$ is orthogonal to Σ, then $h(a) = \langle h, S_a \rangle_b = 0$ for each $a \in \Omega$; thus $h \equiv 0$. In this chapter, we will prove that

Theorem 9.1. *Σ is dense in $A^\infty(\Omega)$.*

To say that Σ is dense in $A^\infty(\Omega)$ means that, given a function $h \in A^\infty(\Omega)$, there is a sequence $H_j \in \Sigma$ such that $H_j(z)$ tends uniformly on $\overline{\Omega}$ to $h(z)$, and each derivative of $H_j(z)$ tends uniformly on $\overline{\Omega}$ to the corresponding derivative of $h(z)$. To prove that Σ is dense in $A^\infty(\Omega)$, we must show that, given $h \in A^\infty(\Omega)$, $\epsilon > 0$, and a positive integer s, there is a function $H \in \Sigma$ such that $|h - H| < \epsilon$ on $\overline{\Omega}$, and $|h^{(m)} - H^{(m)}| < \epsilon$ on $\overline{\Omega}$ for each derivative of order $m \le s$.

If s is a positive integer, let us define the $C^s(b\Omega)$ norm $\|u\|_s$ of a function u defined on the boundary of Ω to be equal to the supremum of the derivatives $|(d^m/dt^m)u(z(t))|$ as the parameter t ranges over its domain and m ranges from zero to s. Although this norm depends on the parameterization $z(t)$ of the boundary, the corresponding space of functions with finite $C^s(b\Omega)$ norm does not.

Although the proof is somewhat technical, the idea is quite simple. In Theorem 3.4, we expressed the boundary values of the Cauchy transform of a smooth function u as an integral,

$$\mathcal{C}u = u - \frac{1}{2\pi i} \iint_{\zeta \in \Omega} \frac{\Psi(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

where Ψ is a function in $C^\infty(\overline{\Omega})$ that vanishes to high order on $b\Omega$. The idea of the proof is to approximate the integral in this expression by a Riemann sum $\sum_{i=1}^N c_i \frac{1}{a_i - z}$. We will see that this can be done in a uniform way as z ranges over the boundary of Ω. Having done this, we take the complex conjugate of the equation and multiply through by $\overline{T(z)}$ to obtain

$$\overline{(\mathcal{C}u)T} \approx \overline{uT} - \sum \bar{c}_i C_{a_i}$$

where C_a denotes the Cauchy kernel. Now, since $\overline{(\mathcal{C}u)T}$ is orthogonal to the Hardy space, it follows that $P(\overline{(\mathcal{C}u)T}) = 0$, and therefore, we may take the Szegő projection of this last formula to obtain

$$P(\overline{uT}) \approx \sum \bar{c}_i P(C_{a_i}).$$

Now, since $P(C_a) = S_a$, we have $P(\overline{uT}) \approx \sum \bar{c}_i S_{a_i}$. If we wanted to approximate a function H in $A^\infty(\Omega)$ by a function in Σ, we would simply choose u so that $H = \overline{uT}$, in which case, the reasoning above would produce an element in Σ that is close to $H = PH = P(\overline{uT})$. That is the idea. Now, here are the details.

If you go back to the proofs of Theorems 3.1 and 4.2 and count derivatives, you will see that we also proved the following result.

Theorem 9.2. *Given a positive integer s, there is a positive integer $n = n(s)$ and a constant $K = K(s)$ such that*

$$\|Pu\|_s \le K\|u\|_n \qquad \text{and} \qquad \|\mathcal{C}u\|_s \le K\|u\|_n$$

for all $u \in C^\infty(b\Omega)$. Consequently, since $C^\infty(b\Omega)$ is dense in $C^n(b\Omega)$, the same inequalities hold for all $u \in C^n(b\Omega)$. In particular, it follows that Pu and $\mathcal{C}u$ are in $C^s(b\Omega)$ whenever $u \in C^n(b\Omega)$.

With some additional effort, the estimates in Theorem 9.2 can be sharpened to allow $n = s + 1$. We will not prove this fact because, for our purposes, it will suffice to know that the estimate holds for some n. With no additional effort, Theorem 9.2 can be sharpened to read as follows.

Given a positive integer s, there is a positive integer $n = n(s)$ and a constant $K = K(s)$ such that

$$\sup\{\left|\frac{d^m}{d^m z}(\mathcal{C}u)(z)\right| : z \in \overline{\Omega},\ 0 \le m \le s\} \le K\|u\|_n,$$

for all $u \in C^n(b\Omega)$. The analogous estimate also holds for the Szegő projection.

To see that this apparently stronger statement is a direct consequence of Theorem 9.2, take a sequence of functions in $C^\infty(b\Omega)$ tending to u in the $C^n(b\Omega)$ norm. Use the maximum principle to see that the boundary estimate in Theorem 9.2 implies the uniform estimate for derivative on

$\overline{\Omega}$ when applied to elements in the sequence. Finally, a simple limiting argument implies the estimate for u.

Theorem 9.2 is the main ingredient in the proof of Theorem 9.1, to which we now return. Let n and $\epsilon > 0$ be given. By Theorem 3.4, we may write the Cauchy transform of a function $u \in C^\infty(b\Omega)$ as $\mathcal{C}u = u - \mathcal{I}$ where, for $z \in b\Omega$,

$$(\mathcal{I})(z) = \frac{1}{2\pi i} \iint_{\zeta \in \Omega} \frac{\Psi(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}$$

and Ψ is a function in $C^\infty(\overline{\Omega})$ that vanishes to order n on the boundary of Ω. Since Ψ vanishes to order n on the boundary, we may think of Ψ as a function in $C^n(\mathbb{C})$ by extending it to be zero outside Ω. We now claim that, given $\delta > 0$, we can find a function Ψ_δ in $C^n(\mathbb{C})$ that has *compact support* in Ω such that for any derivative D^α of order n or less, we have $|D^\alpha(\Psi_\delta - \Psi)| < \delta$ on $\overline{\Omega}$. To construct such a Ψ_δ, we take a partition of unity subordinate to a covering of $\overline{\Omega}$ by small discs, thereby enabling us to assume that Ψ is a function supported in a small disc $D_r(z_0)$ where $z_0 \in b\Omega$. The complex number $w_0 = -iT(z_0)$ represents the outward pointing normal vector to $b\Omega$ at z_0. Now, if the disc is small enough, the translated function $\Psi_\delta(z) = \Psi(z + \lambda w_0)$ will be well defined on Ω and will have the properties we seek provided λ is chosen small enough. The claim is proved.

Define

$$(\mathcal{I}_\delta)(z) = \frac{1}{2\pi i} \iint_{\zeta \in \Omega} \frac{\Psi_\delta(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

We next claim that it is possible to choose δ so small that the modulus of $\mathcal{I} - \mathcal{I}_\delta$ and the moduli of the derivatives of $\mathcal{I} - \mathcal{I}_\delta$ of order n or less are less than $\epsilon/2$ on $\overline{\Omega}$. In fact, to see this, we make the change of variables in the integral as we did in the proof of Theorem 2.2. This allows us to write

$$(\mathcal{I})(z) - (\mathcal{I}_\delta)(z) = -\frac{1}{2\pi i} \iint_{\zeta \in \mathbb{C}} \frac{\Psi(z - \zeta) - \Psi_\delta(z - \zeta)}{\zeta} \, d\zeta \wedge d\bar{\zeta},$$

and by differentiating under the integral, and using the fact that the kernel $1/\zeta$ is locally integrable, our claim follows.

Now, because Ψ_δ has compact support, for $z \in b\Omega$, we may approximate the integral defining $\mathcal{I}_\delta$ by a (finite) Riemann sum

$$\mathcal{S}(z) = \frac{1}{2\pi i} \sum c_i \frac{1}{a_i - z}$$

in such a way that $\|\mathcal{S} - \mathcal{I}_\delta\|_n < \epsilon/2$.

We have now shown that $\|u - \mathcal{C}u - \mathcal{S}\|_n < \epsilon$ where ϵ can be taken to be arbitrarily small. If we now multiply $u - \mathcal{C}u - \mathcal{S}$ by T and take the complex conjugate, we see that the $C^n(b\Omega)$ norm of

$$\overline{Tu} - \overline{T\mathcal{C}u} - \sum \bar{c}_i C_{a_i}$$

can be made arbitrarily small. Next, we take the Szegő projection of this function. Using the facts that $P(\overline{T\mathcal{C}u}\,) = 0$ and $S(z,a) = (PC_a)(z)$, we see that by choosing n sufficiently large and ϵ sufficiently small, the $C^s(b\Omega)$ norm of

$$P(\overline{Tu}) - \sum \bar{c}_i S(\cdot, a_i)$$

can be made arbitrarily small by virtue of Theorem 9.2. To finish the proof of Theorem 9.1, we need only note that a function h in $A^\infty(\Omega)$ can be written as $\overline{Tu}$ where $u = \overline{Th}$. Hence $h = Ph = P(\overline{Tu})$ can be approximated in $C^s(b\Omega)$ norm by functions in Σ.

The following corollaries will be seen to be simple consequences of Theorem 9.1.

Corollary 9.1. *Given a point w_0 in the boundary of Ω, the function $h(z) = S(z, w_0)$ cannot be identically zero as a function of z on Ω. Also, the function $H(z) = L(z, w_0)$ cannot be identically zero as a function of z on Ω.*

For $a \in \Omega$, let $L_a(z) = L(z,a)$, and let Λ denote the linear span of $\{L_a(z) : a \in \Omega\}$.

Corollary 9.2. *Λ is dense in $C^\infty(b\Omega) \cap H^2(b\Omega)^\perp$ in the sense of approximation in $C^s(b\Omega)$ norms for arbitrarily high s.*

To prove Corollary 9.1, note that, according to Theorem 9.1, the function $h(z) \equiv 1$ can be approximated uniformly on $b\Omega$ by functions in Σ. If $S(z, w_0)$ were identically zero in z, then every function in Σ would vanish at w_0. Thus, it would be impossible to approximate $h \equiv 1$ uniformly near w_0, contradicting Theorem 9.1. The statement about the nonvanishing of $L(z, w_0)$ now follows from that of $S(z, w_0)$ and the identity $iS(z, w_0) = L(w_0, z)T(w_0)$, which holds for all $z \in \Omega$.

Corollary 9.2 follows from the density of Σ, identity (7.1), and the orthogonal decomposition of $L^2(b\Omega)$. Indeed, if $\overline{HT}$ is in $C^\infty(b\Omega) \cap H^2(b\Omega)^\perp$, we may approximate H by elements s_j of Σ. Now $\overline{HT}$ is approximated by the sequence of functions $\overline{s_j T}$ which belong to Λ by virtue of identity (7.1) rewritten in the form $\overline{S(z,a)T(z)} = -iL(z,a)$.

Recall that $L(z,a) = -L(a,z)$ for a and z in Ω. Hence, if $L(b,a)$ were to vanish for some values of a and b in Ω, $a \neq b$, it would follow that $L(a,b) = 0$ too. Consequently, $L(z,a)L(z,b)$ would be a holomorphic function of z on *all* of Ω that is in $C^\infty(\overline{\Omega})$. To be precise, the zero of $L(z,a)$ at $z=b$ would cancel out the pole of $L(z,b)$ at $z=b$ and the zero of $L(z,b)$ at $z=a$ would cancel out the pole of $L(z,a)$ at $z=a$. This observation allows us to give a simple proof of the nonvanishing of the Garabedian kernel in simply connected domains.

Theorem 9.3. *If Ω is a bounded simply connected domain with C^∞ smooth boundary, then $L(z,a) \neq 0$ for all $z \in \overline{\Omega} - \{a\}$.*

It is a general fact that $L(a,b) \neq 0$ for all $a, b \in \overline{\Omega}$, $a \neq b$, even if Ω is a smooth *multiply connected* domain. We will prove this more general result in Chapter 13.

Proof. Suppose $L(a,b) = 0$ for some a and b in Ω, $a \neq b$. Let $\mathcal{L}(z) = L(z,a)L(z,b)$ and let $\mathcal{S}(z) = S(z,a)S(z,b)$. Identity (7.1) yields that,

$$-\mathcal{L}(z)T(z) = \overline{\mathcal{S}(z)T(z)}, \qquad z \in b\Omega. \tag{9.1}$$

As remarked above, $\mathcal{L}$ is in $A^\infty(\Omega)$. So is $\mathcal{S}$. Hence, by Theorem 4.3, formula (9.1) implies that $\mathcal{L}T$ is orthogonal to $H^2(b\Omega)$ and also orthogonal to the space of conjugates of functions in $H^2(b\Omega)$. In a simply connected domain, this forces us to conclude that $\mathcal{L} \equiv 0$. Indeed, it follows from this orthogonality that, for any $h \in A^\infty(\Omega)$, we have

$$0 = \int_{z\in b\Omega} \mathcal{L}\, T\, \overline{h}\, ds = \int_{z\in b\Omega} \mathcal{L}\, \overline{h}\, dz.$$

Now, since $d\bar{z} \wedge dz = 2i\, dx \wedge dy$, it follows from the complex Green's formula that

$$\iint_\Omega \mathcal{L}\, \overline{h'}\, dx \wedge dy = 0$$

for all $h \in A^\infty(\Omega)$. Since $\mathcal{L}$ is in $A^\infty(\Omega)$, and since Ω is simply connected, there is a function $h(z)$ in $A^\infty(\Omega)$ such that $h'(z) = \mathcal{L}(z)$. Hence, it follows that $\iint_\Omega |\mathcal{L}|^2\, dx \wedge dy = 0$ and therefore, that $\mathcal{L} \equiv 0$, i.e., that $L(z,a)L(z,b)$ is identically zero. This implies that, either $L(z,a) \equiv 0$, or $L(z,b) \equiv 0$, and it is impossible for either of these functions to vanish identically in $z \in b\Omega$ because they have poles at a and b, respectively.

We have shown that $L(z,a)$ is nonvanishing for all $z \in \Omega - \{a\}$. To see that $L(z_0,a) \neq 0$ for $z_0 \in b\Omega$, let z_j be a sequence of points in Ω that tend to z_0. Let ϵ be small enough so that the disc $D_{2\epsilon}(a)$ about a is compactly contained in Ω, and let $\overline{\Omega}_\epsilon = \overline{\Omega} - D_{2\epsilon}(a)$. The functions $H_j(w) = L(z_j, w)$ are holomorphic and nonvanishing on $D_\epsilon(a)$ as functions of w if j is large. We claim that these functions converge uniformly on $D_\epsilon(a)$ to $H_0(w) = L(z_0, w)$ as $j \to \infty$. Assuming this claim, it then follows by Hurwitz's Theorem, that either the limit function is identically zero, or never zero. By Corollary 9.1, $L(z_0, w)$ cannot be identically zero. Thus, $L(z_0, w)$ is never zero, and we conclude that $L(z_0, a) \neq 0$. Therefore, to finish the proof, it will suffice to verify the claim by showing that $L(z, w)$ is continuous for (z, w) in $\overline{\Omega}_\epsilon \times D_\epsilon(a)$. Recall that $L_w = P^\perp G_w$ where $G_w(z) = [2\pi(z - w)]^{-1}$. Hence $L_w = G_w - P(G_w)$. Theorem 9.2 shows that $P(G_w)(z)$ is continuous on $(z, w) \in (\overline{\Omega} \times D_\epsilon(a))$. It is clear that $G_w(z)$ is continuous on $(z, w) \in (\overline{\Omega}_\epsilon \times D_\epsilon(a))$. Hence, $L(z, w)$ is continuous on $(z, w) \in (\overline{\Omega}_\epsilon \times D_\epsilon(a))$, and the proof of Theorem 9.3 is complete. □

We are now in a position to prove Theorem 8.2. In fact, the proof we are about to give includes an existence proof for the Riemann mapping function associated to a point a in a bounded simply connected domain Ω with C^∞ smooth boundary.

Proof of Theorem 8.2. Theorem 9.3 says that L_a is nonvanishing on $b\Omega$. It follows from (7.1) that S_a is nonvanishing on $b\Omega$ too. Hence, we are justified in using the argument principle to count the zeroes of S_a in Ω, and to count the pole of L_a in Ω. The main tool we will use to relate these numbers is identity (7.1). Let $\Delta\arg h$ denote the net increase in the argument of a nonvanishing function $h(z)$ defined on $b\Omega$ as z traces out the boundary of Ω in the standard sense. Note that $\Delta\arg \overline{h} = -\Delta\arg h$ and that $\Delta\arg T = 2\pi$.

It follows from (7.1) that $-\Delta\arg S_a = (\Delta\arg L_a) + 2\pi$. The increase of $\arg L_a$ around $b\Omega$ is -2π because L_a has a single simple pole at a, and no zeroes in $\overline{\Omega}$. Hence, $\Delta\arg S_a = 0$ and we conclude that S_a has no zeroes in Ω. In fact, S_a is nonvanishing on $\overline{\Omega}$ because, as mentioned above, S_a does not vanish on $b\Omega$.

We can now say that the map given by $f = S_a/L_a$ is in $A^\infty(\Omega)$, and that $|f| = 1$ on $b\Omega$. Note that f has a simple zero at a due to the pole of L_a at a and the fact that $S(a, a) \neq 0$. Since $f(a) = 0$, the maximum principle yields that $|f| < 1$ on Ω, i.e., that f maps Ω into the unit disc. For w in the unit disc, consider the integral

$$N(w) = \frac{1}{2\pi i} \int_{b\Omega} \frac{f'(z)}{f(z) - w}\, dz.$$

The argument principle says that $N(w)$ is equal to the number of points $z \in \Omega$ satisfying $f(z) = w$. Since $N(w)$ is integer valued and holomorphic on the unit disc, and since $N(0) = 1$, it follows that $N(w) \equiv 1$. We conclude that f is a one-to-one map of Ω onto the unit disc.

To see that f' cannot vanish on $b\Omega$, we use the Harnack inequality ([Ah, p. 243]). Suppose $z_0 \in b\Omega$. We may assume that $f(z_0) = 1$. Let $h(z) = 1 - \text{Re } z$. This function is harmonic and assumes its minimum value of zero on the closed unit disc at $z = 1$. The function $h \circ f$ is harmonic on Ω, is in $C^\infty(\overline{\Omega})$, and assumes its minimum value of zero on $\overline{\Omega}$ at z_0. Let $D_r(w_0)$ be a small disc of radius r contained in Ω that is internally tangent to the boundary of Ω at z_0. If we now apply Harnack's inequality to the positive harmonic function $h \circ f$ on $D_r(w_0)$, we obtain

$$\frac{r - |z - w_0|}{r + |z - w_0|} h(f(w_0)) \leq h(f(z)).$$

But $h(f(w_0)) = c$ is a positive constant. Hence,

$$\frac{h(f(z)) - h(f(z_0))}{r - |z - w_0|} \geq \frac{c}{r + |z - w_0|} \geq \frac{c}{2r}.$$

By letting z approach z_0 along the inward pointing normal to $b\Omega$ at z_0, we deduce that the normal derivative of $h \circ f$ at z_0 is nonzero. Hence, $f'(z_0)$ cannot vanish. The proof of Theorem 8.2 is complete. □

We remark that the argument just given to show that f' is nonvanishing on the boundary contains a proof of the following classical result known as the *Hopf lemma*.

Theorem 9.4. *Suppose that Ω is a bounded domain with C^∞ smooth boundary and that ϕ is a harmonic function on Ω that is in $C^1(\overline{\Omega})$. If the maximum value of ϕ on $\overline{\Omega}$ is attained at a boundary point z_0, then the normal derivative of ϕ at z_0 is strictly positive.*

We mention one last result. We proved it above in the existence proof of the Riemann map.

Theorem 9.5. *If Ω is a bounded simply connected domain with C^∞ smooth boundary and $a \in \Omega$, then $S(z, a) \neq 0$ for all $z \in \overline{\Omega}$.*

10

Solution of the Dirichlet problem in simply connected domains

Given a continuous function φ on $b\Omega$, the classical Dirichlet problem is to find a harmonic function u on Ω that extends continuously to $b\Omega$ and that agrees with φ on $b\Omega$. In this chapter, we will relate the solution of this problem to the Szegő projection. In fact, we will prove that the solution to the analogous problem in the C^∞ setting exists and is well behaved. We will then use the C^∞ result to solve the classical problem.

Theorem 10.1. *Suppose Ω is a bounded simply connected domain with C^∞ smooth boundary and suppose φ is a function in $C^\infty(b\Omega)$. Let $a \in \Omega$ and let $S_a(z) = S(z,a)$ and $L_a(z) = L(z,a)$. Then, the function $u = h + \overline{H}$, where h and H are holomorphic functions in $A^\infty(\Omega)$ given by*

$$h = \frac{P(S_a\,\varphi)}{S_a} \qquad \textit{and} \qquad H = \frac{P(L_a\,\overline{\varphi})}{L_a},$$

solves the Dirichlet problem for φ. Note that it follows that $u \in C^\infty(\overline{\Omega})$.

We remark that, since every harmonic function on a simply connected domain can be written as the real part of a holomorphic function, it is possible to decompose any harmonic function as $g + \overline{G}$. If a harmonic function u is decomposed in two ways as $u = g_1 + \overline{G_1} = g_2 + \overline{G_2}$, then $g_1 - g_2 = \overline{G_2} - \overline{G_1}$ is both holomorphic and antiholomorphic, hence, constant. Thus g and G in the decomposition $u = g + \overline{G}$ are uniquely determined up to additive constants. The functions h and H in the theorem are therefore uniquely determined by the condition that $H(a) = 0$, which follows from the fact that L_a has a pole at a. The maximum principle implies that the solution to the Dirichlet problem is unique. Hence, although the functions h and H depend on the choice of a, the solution to the Dirichlet problem does not.

Note that it follows from Theorem 10.1 and the preceding remarks that if a harmonic function on Ω that is in $C^\infty(\overline{\Omega})$ is decomposed as $g + \overline{G}$ where g and G are holomorphic on Ω, then it must be that g and G are in $A^\infty(\Omega)$.

It is easy to deduce from Theorem 10.1 that the solution to the classical Dirichlet problem exists. Indeed, if φ is a continuous function on

$b\Omega$, let φ_j be a sequence of functions in $C^\infty(b\Omega)$ that converge uniformly on $b\Omega$ to φ. The Maximum Principle can now be used to see that the solutions u_j to the Dirichlet problems corresponding to φ_j converge uniformly on $\overline{\Omega}$ to a function u that is harmonic on Ω, continuous on $\overline{\Omega}$, and that assumes φ as its boundary values. This same limiting argument reveals that the formula in Theorem 10.1 expresses the solution to the Dirichlet problem even when φ is merely assumed to be continuous on $b\Omega$. It must be pointed out, however, that in this case, the functions h and H need not extend continuously to the boundary, even though the solution $h + \overline{H}$ does.

Proof of Theorem 10.1. The function $S_a\varphi$ has an orthogonal decomposition $S_a\varphi = g + \overline{GT}$ where $g = P(S_a\varphi)$ and $\overline{GT} = P^\perp(S_a\varphi)$. Identity (7.1) yields that $\overline{T} = -iS_a/\overline{L_a}$. Hence, $S_a\varphi = g - i\,\overline{G}\,S_a/\overline{L_a}$, and upon dividing this equation by S_a, we obtain

$$\varphi = \frac{g}{S_a} - i\frac{\overline{G}}{\overline{L_a}}.$$

By Theorem 9.5, S_a does not vanish on $\overline{\Omega}$, and by Theorem 9.3, L_a does not vanish on $\overline{\Omega} - \{a\}$. Hence, the first term in this decomposition is seen to be in $A^\infty(\Omega)$ and the second term is the conjugate of a function in $A^\infty(\Omega)$. It follows that the sum is a harmonic function in $C^\infty(\overline{\Omega})$ that agrees with φ on $b\Omega$ and we have found a solution to the Dirichlet problem.

To finish the proof of the theorem, we must show that $iG = P(L_a\,\overline{\varphi}\,)$. Formula (4.4) says that $P^\perp v = \overline{T}\,\overline{P(\overline{vT})}$. Thus, $G = P(\overline{S_a\varphi T})$. But, identity (7.1) implies that $\overline{S_a T} = -iL_a$. Therefore, $G = -iP(L_a\overline{\varphi})$, and the proof is finished. □

The proof just given is somewhat sneaky. However, there is an easy way to see that the formulas should be true. Suppose u is a harmonic function on Ω in $C^\infty(\overline{\Omega})$ and $u = h + \overline{H}$ where h and H are in $A^\infty(\Omega)$, and $H(a) = 0$. Then $S_a u = S_a h + \overline{H}\,S_a$. This is actually an orthogonal decomposition of $S_a u$ because, by (7.1), $S_a = i\,\overline{L_a T}$, and so $\overline{H}\,S_a = i\,\overline{H L_a T}$; the zero of H at a cancels the pole of L_a at a, and hence this term is in $H^2(b\Omega)^\perp$. Thus, $P(S_a u) = S_a h$. A similar argument yields the formula for H. We could not use this more straightforward approach in our proof because we could not say in advance that, for a given $\varphi \in C^\infty(b\Omega)$, a solution u exists to the Dirichlet problem. Even if we knew a solution existed, we could not say that h and H in the decomposition for u must be in $A^\infty(\Omega)$. In fact, we could not even say that h and H must be in $H^2(b\Omega)$. However, these facts were byproducts of the sneaky proof above.

Theorem 10.1 can be localized. The explicit form of the solution operator to the Dirichlet problem given in Theorem 10.1 together with Theorem 4.4 yield the following result.

Theorem 10.2. *Suppose u is a harmonic function on Ω that extends continuously to $\overline{\Omega}$. If the boundary values of u on an open arc $\Gamma \subset b\Omega$ are C^∞ smooth there, then all partial derivatives of u extend continuously to $\Omega \cup \Gamma$.*

The formulas in Theorem 10.1 yield interesting results if we think of φ as merely being in $L^2(b\Omega)$. Indeed, if $\varphi \in L^2(b\Omega)$, then the decomposition $\varphi = h + \overline{H}$, where $h = P(S_a\varphi)/S_a$ and $H = P(L_a\overline{\varphi})/L_a$, is valid on the boundary of Ω. Consider the harmonic function u on Ω given by $u = h + \overline{H}$. If z is a boundary point of Ω, let $u_\epsilon(z) = u(z + i\epsilon T(z))$. Because of Theorem 6.3, we may assert that u_ϵ tends to u in $L^2(b\Omega)$ as ϵ tends to zero. Hence, it is possible to solve the Dirichlet problem starting with $L^2(b\Omega)$ boundary data, thereby obtaining a harmonic function that has the data as its boundary values in an L^2 sense. Another interesting consequence of the kind of reasoning above is an L^2 maximum principle. To be specific, if a harmonic function u has vanishing boundary values in the sense that u_ϵ, as defined above, tends to zero in $L^2(b\Omega)$, then u must be zero in the interior.

If φ is a continuous function on $b\Omega$, the *Poisson extension* of φ, $\mathcal{E}\varphi$, is defined to be equal to the harmonic function u on Ω that solves the Dirichlet problem with boundary data φ. We have shown that the Poisson extension operator is related to the Szegő projection in simply connected domains via the identity in Theorem 10.1, and that $\mathcal{E}$ maps $C^\infty(b\Omega)$ into $C^\infty(\overline{\Omega})$. When the integrals for the Szegő projections are written out in that identity, we obtain the *Poisson kernel* for a simply connected domain. Indeed,

$$(\mathcal{E}\varphi)(z) = \int_{w \in b\Omega} p(z,w)\varphi(w)\, ds$$

where, for $z \in \Omega$ and $w \in b\Omega$, the Poisson kernel $p(z,w)$ is given by

$$p(z,w) = \frac{S(z,w)S(w,a)}{S(z,a)} + \frac{\overline{S(z,w)L(w,a)}}{\overline{L(z,a)}}. \tag{10.1}$$

This formula for the Poisson kernel seems to depend on the choice of $a \in \Omega$. However, if $p_1(z,w)$ and $p_2(z,w)$ were the Poisson kernels corresponding to choosing $a = a_1$ and $a = a_2$, respectively, then, by the uniqueness of the solution to the Dirichlet problem, $q(w) = p_1(z,w) - p_2(z,w)$ would be orthogonal to every continuous function on $b\Omega$, and consequently would have to be identically zero. Hence, the kernel $p(z,w)$ does not depend on the point a. It is interesting to note that we can let $a = z$ to see that

$$p(z,w) = \frac{|S(w,z)|^2}{S(z,z)}. \tag{10.2}$$

The function on the right hand side of this equality is known as the Poisson-Szegő kernel. We have shown that, in a simply connected domain, the Poisson kernel and the Poisson-Szegő kernel coincide.

The Poisson kernel of a domain Ω has many of the properties of the classical Poisson kernel on the unit disc. Most of these key properties can be read off from formulas (10.1) and (10.2). For example, $p(z,w)$ is strictly positive on $\Omega \times b\Omega$ by (10.2) and the fact that the Szegő kernel is nonvanishing in a simply connected domain. It is also clear that $\int_{w\in b\Omega} p(z,w)\,ds = 1$ for all $z \in \Omega$ because the function $u \equiv 1$ is the harmonic extension of 1. Formula (10.1) shows that $p(z,w)$ is harmonic in z on Ω for fixed $w \in b\Omega$.

There is one other well known property of the Poisson kernel for the unit disc that we would like to prove for $p(z,w)$, but we will have to save the proof for later. It will be proved in Chapter 26 that, given $w_0 \in b\Omega$ and $\delta > 0$, $p(z,w)$ tends to zero uniformly in w on the set $b\Omega - D_\delta(w_0)$ as z tends to the boundary while staying in the set $\Omega \cap D_{\delta/2}(w_0)$.

11

The case of real analytic boundary

A domain in the plane is said to have *real analytic boundary* if its boundary can be (locally) parameterized by a function $z(t) = x(t) + iy(t)$ where the real valued functions $x(t)$ and $y(t)$ are equal to their (real) Taylor series expansions in $(t - t_0)$ in a neighborhood of each point t_0 in the parameter space. A function $v(x, y)$ will be said to be real analytic on an open set in the plane if it can be expanded in a power series $v(x, y) = \sum a_{nm}(x - x_0)^n(y - y_0)^m$ that converges on a neighborhood of each point (x_0, y_0) in the open set. Note that harmonic functions are real analytic because they are locally the real part of holomorphic functions.

In this chapter, we will indicate how the arguments we have given in the C^∞ setting can be modified to give analogous results for domains with real analytic boundary.

Theorem 11.1. *Suppose Ω is a bounded domain with real analytic boundary. Suppose that v is a function in $C^\infty(\overline{\Omega})$ that extends to be defined on a neighborhood of $\overline{\Omega}$ in such a way that the extension is real analytic on a neighborhood of $b\Omega$. Then the solution u,*

$$u(z) = \frac{1}{2\pi i} \iint_\Omega \frac{v(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta},$$

to the equation, $\partial u/\partial \bar{z} = v$, also extends to be real analytic in a neighborhood of $b\Omega$.

Since the complex conjugate of $\partial u/\partial \bar{z}$ is equal to $\partial \bar{u}/\partial z$, Theorem 11.1 implies a similar statement about the existence of nice solutions to the equation $\partial u/\partial z = v$.

As in the C^∞ case, the theorem is a simple consequence of the Cauchy Integral formula and a lemma.

Lemma 11.1. *Suppose that Ω is a bounded domain with real analytic boundary and that $v \in C^\infty(\overline{\Omega})$ extends to be real analytic in a neighborhood of $b\Omega$. Then, there exists a function $\Phi \in C^\infty(\overline{\Omega})$ that vanishes on the boundary of Ω such that $\partial\Phi/\partial\bar{z} = v$ near $b\Omega$, i.e., such that $\partial\Phi/\partial\bar{z} - v$ is in $C_0^\infty(\Omega)$.*

Proof of the lemma. We give a complete, elementary, and self contained proof of this lemma in Appendix A. Here, we show how the lemma can be seen to be a direct consequence of the Cauchy-Kovalevski theorem. (see Folland [Fo] for a proof of this famous theorem). Indeed, every curve in the plane is noncharacteristic for the Laplace operator Δ. Hence, we may solve the Cauchy problem, $\Delta\psi = v$ near the boundary of Ω with Cauchy conditions $\psi = 0$ on $b\Omega$ and $\nabla\psi = 0$ on $b\Omega$. The solution ψ will be defined and real analytic on a neighborhood of $b\Omega$. We now claim that the function $\varphi = 4(\partial/\partial z)\psi$ solves our Cauchy problem near $b\Omega$. Indeed, since $\Delta = 4(\partial/\partial\bar{z})(\partial/\partial z)$, and since $\Delta\psi = v$, it follows that $\partial\varphi/\partial\bar{z} = v$. Furthermore, because ψ and $\nabla\psi$ vanish on $b\Omega$, it follows that $\varphi = 0$ on $b\Omega$. To extend our solution φ to all of Ω, we simply use a C^∞ cutoff function χ that is compactly supported inside the set where ψ is defined and real analytic, and that is equal to one on a small neighborhood of $b\Omega$. Now the function Φ, which is defined to be zero where χ is zero and which is defined to be equal to $\chi\varphi$ on the support of χ, is a function with the properties we seek. $\square$

When the boundary of a domain is real analytic, we may define what it means for a function to be real analytic on the boundary. A function u defined on $b\Omega$ is said to be real analytic if $u(z(t))$ is a real analytic function of t when $z(t)$ is a real analytic parameterizing function for $b\Omega$. It is an easy exercise to see that u is real analytic on $b\Omega$ if and only if $u(z(t))$ is real analytic for a *single* real analytic parameterization $z(t)$. We will use the symbol $C^\omega(b\Omega)$ to denote the space of real analytic functions on $b\Omega$. It is an elementary fact that $C^\omega(b\Omega)$ coincides with the space of continuous functions on $b\Omega$ that are restrictions to $b\Omega$ of functions that are defined and real analytic in a neighborhood of $b\Omega$. In fact, we will now show that $C^\omega(b\Omega)$ is equal to the set of functions on $b\Omega$ that are restrictions to $b\Omega$ of functions that are holomorphic in a neighborhood of $b\Omega$. To see this, suppose that $z(t) = \sum c_n(t-t_0)^n$ is a real analytic parameterization of $b\Omega$ with t near t_0. Consider the function $f(\zeta) = \sum c_n(\zeta - t_0)^n$ which is defined and holomorphic near $\zeta = t_0$. Since $z'(t_0) \neq 0$, it follows that $f'(\zeta) \neq 0$ near $\zeta = t_0$. Hence, we may define a holomorphic inverse $F(z)$ to f near $z(t_0)$. This function has the property that it maps the boundary curve of Ω near $z(t_0)$ one-to-one onto a segment in the real axis of the complex plane. If u is in $C^\omega(b\Omega)$, then $u(z(t))$ is real analytic near $t = t_0$. Let $U(\zeta)$ be the holomorphic function defined on a neighborhood of $\zeta = t_0$ obtained by replacing the real variable t in the power series expansion for $u(z(t))$ by a complex variable ζ. Now $U(F(z))$ is a holomorphic function defined in a neighborhood of $z(t_0)$ whose restriction to the boundary of Ω near $z(t_0)$ agrees with u.

This is an opportune moment to prove the general version of the Schwarz reflection principle because the argument in the last paragraph is at the heart of its proof. Suppose f is a holomorphic function defined on one side of a real analytic curve γ_1 that extends continuously up to γ_1 and that maps γ_1 into another real analytic curve γ_2. The Schwarz reflection principle asserts that f must extend holomorphically past γ_1. To prove this, we restrict our attention to a small neighborhood of a point in γ_1. As in the last paragraph, we may construct holomorphic functions F_1 and F_2 such that F_j is holomorphic in a neighborhood of γ_j and maps γ_j into a segment of the real axis in the complex plane, $j = 1, 2$. By restricting to a small enough neighborhood, we may guarantee that the functions F_j have nonvanishing derivatives near γ_j. We may also suppose that F_1 maps the side of γ_1 on which f is defined into the upper half plane. Now consider the map $H = F_2 \circ f \circ F_1^{-1}$. It is defined on a subregion of the upper half plane, it extends continuously to a segment in the real axis, and it maps this segment in the real axis into the real axis. Hence, we may apply the classical Schwarz reflection principle (see Ahlfors [Ah, p. 172]) to see that H extends holomorphically across the real axis. Now $f = F_2^{-1} \circ H \circ F_1$ is seen to extend holomorphically through γ_1.

Now that we have proved the reflection principle, it is an opportune moment to mention reflection functions. Later in this book, when we consider the boundary behavior of the Szegő and Garabedian kernels of domains with real analytic boundaries in detail, we will need to know of the existence of *antiholomorphic reflection functions.* Suppose that γ is a real analytic curve. An antiholomorphic reflection function $R(z)$ for γ is a function satisfying the following properties. It is defined and antiholomorphic on a neighborhood of γ, $R(R(z)) = z$ for all z, R locally maps one side of γ to the other side, and $R(w) = w$ for w on γ. To see that such functions exist, it is enough to check that they exist locally because analytic continuation may be used to obtain global reflections from local ones. To do the local construction, let $F(z)$ be a holomorphic function as we produced above which is defined on a neighborhood of a point in γ and which maps γ into the real axis. The function $z \mapsto \bar{z}$ is an antiholomorphic reflection for the real axis. Hence, it follows that $R(z) = F^{-1}(\,\overline{F(z)}\,)$ defines an antiholomorphic reflection for γ.

We will use reflection functions later when we show that kernel functions extend past the boundary in both variables simultaneously. At the moment, we will be satisfied to show only that the kernels extend past the boundary in one variable when the other variable is held fixed at a point in the interior. This result will follow easily from the next theorem.

Theorem 11.2. *Suppose Ω is a bounded domain with real analytic boundary. The Cauchy transform maps $C^\omega(b\Omega)$ into itself. So does the*

Szegő projection. Hence, for a fixed point $a \in \Omega$, the kernel functions $S(z,a)$ and $L(z,a)$ both extend holomorphically past the boundary of Ω in the z variable.

The proof of Theorem 11.2 mirrors the proofs of Theorems 3.1 and 4.2. One of the key steps is to prove that the Kerzman-Stein kernel $A(z,w)$ is in $C^\omega(b\Omega \times b\Omega)$, but this is routine.

The Szegő kernel is the Szegő projection of the Cauchy kernel, which is obviously a function in $C^\omega(b\Omega)$. The Garabedian kernel is equal to $P^\perp G_a$ where $G_a(z) = (2\pi)^{-1}(z-a)^{-1}$. Hence, it follows from Theorem 11.2 that the kernels $S(z,a)$ and $L(z,a)$ associated to a domain with real analytic boundary extend holomorphically past the boundary as functions of z when $a \in \Omega$ is fixed.

Another interesting consequence of Theorem 11.2 is that if a holomorphic function that extends continuously to the boundary of a domain with real analytic boundary has real analytic boundary values near a given boundary point, then that function must extend holomorphically past the boundary near the point.

All of the rest of the theorems we have proved in the C^∞ setting can now be routinely generalized to the C^ω case. We mention one of them that is particularly interesting and useful. We note that we are using the elementary fact that a real analytic function on a connected open set that is harmonic on a small open subset must be harmonic on the whole set.

Theorem 11.3. *Suppose Ω is a bounded simply connected domain with real analytic boundary and suppose φ is a function in $C^\omega(b\Omega)$. The solution to the Dirichlet problem with boundary data φ extends to be defined on a neighborhood of $\overline{\Omega}$ in such a way as to be harmonic there.*